

NUMERICAL ANALYSIS PRELIMINARY EXAM

AUGUST 2009

Do as many problems and parts of problems as time allows.

1. Find the leading order terms (in both Δx and Δt) of the local truncation error of the explicit scheme

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{(U_{j+1}^n - U_j^n) p_{j+1/2} - (U_j^n - U_{j-1}^n) p_{j-1/2}}{(\Delta x)^2}$$

for the differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left(p(x) \frac{\partial u}{\partial x} \right)$$

on the region $0 < x < 1, t > 0$. Here $U_j^n \approx u(n\Delta t, j\Delta x), p_{j\pm 1/2} = p((j \pm 1/2)\Delta x)$, assuming $u(t, x)$ and $p(x)$ are sufficiently smooth.

2. Consider the upwind scheme

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + a \frac{U_j^n - U_{j-1}^n}{\Delta x} = 0$$

applied to the one-way wave equation

$$u_t + au_x = 0, \quad a > 0,$$

with $\nu = a \frac{\Delta t}{\Delta x}$.

- (a) Determine the CFL condition of the scheme.
- (b) Use Von Neumann analysis to determine the range of ν for which the scheme is stable.

3. Consider the method

$$U_j^{n+1} = U_j^n - \frac{ak}{2h} (U_j^n - U_{j-1}^n + U_j^{n+1} - U_{j-1}^{n+1})$$

for the advection equation $u_t + au_x = 0$ on $0 \leq x \leq 1$ with periodic boundary conditions. Here k is the time step size and h is the spacial grid size.

- (a) This method can be viewed as the trapezoidal method applied to an ODE system $U'(t) = AU(t)$ arising from a method of lines discretization of the original PDE. Here we assume the spacial grid contains m subintervals, i.e., $h = 1/m, U_0(t) = U_m(t)$ from the periodic boundary conditions, and the unknown vector is $U(t) = [U_1(t), U_2(t), \dots, U_m(t)]'$. Find the $m \times m$ matrix A and its eigenvalues. [*Hint: Pay attention to the boundary conditions when deriving A . The p -th ($1 \leq p \leq m$) eigenvalue of A has the corresponding eigenvector $u_j^p = e^{2\pi i p j h}, 1 \leq j \leq m$].*
- (b) Give the absolute stability region of the trapezoidal method for solving ODE $u' = \lambda u$ in terms of the parameter $z = k\lambda$, where k is the time step. [*You don't have to prove it*].
- (c) Suppose we keep the Courant number $\nu = ak/h$ fixed as $k, h \rightarrow 0$ and $a > 0$. For what range of the ν will the method be stable? Justify your answer using the eigenvalues of A obtained in (a) and the absolute stability region of the trapezoidal method given in (b).

4. Consider the axisymmetric problem governed by the following differential equation

$$-\frac{1}{r} \frac{d}{dr} \left[a(r) \frac{du}{dr} \right] = f(r), \quad \text{for } 0 \leq r \leq L, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq z \leq 1,$$

with boundary conditions

$$\left. \frac{du}{dr} \right|_{r=0} = 0, \quad \left. a(r) \frac{du}{dr} + \beta(u - u_\infty) \right|_{r=L} = 0.$$

Here β, u_∞ are constants. Note that the domain is three dimension, but the solution u only depends on the radius r . Recall the weak formulation of this problem is to find $u(r)$ such that

$$B(w, u) = l(w), \quad \forall w \in W.$$

Find the bilinear functional $B(w, u)$ and linear functional $l(w)$. You need to integrate over the domain in cylindrical coordinates, and assume the weight function w is a function of r only. Does the weight function w have to satisfy certain boundary conditions? Why?

5. Solve the Laplace equation $-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$ for the unit square domain and boundary conditions given in the figure with one linear rectangular element. Show the following steps: weak formulation, element equation (assembly is not necessary since there is only one element), condensed equation resulted from imposing boundary conditions. The final answer should be the value of approximate solution u_h at points $(1, 1)$ and $(0, 1)$.

