Ph.D. Comprehensive Exam: Numerical Analysis

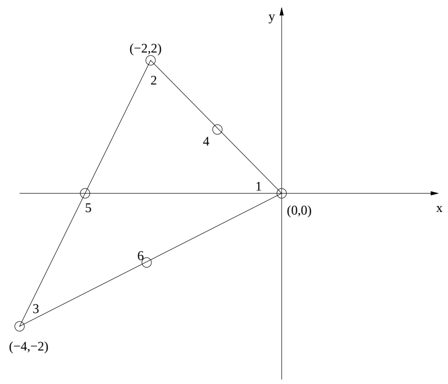
August 2011

1. Show that the finite difference scheme
$$(1+\frac{1}{12}\delta_x^2)(U_j^{n+1}-U_j^n) = \frac{1}{2}\mu\delta_x^2(U_j^{n+1}+U_j^n) + \frac{1}{2}\Delta t\left[f_j^{n+1} + (1+\frac{1}{6}\delta_x^2)f_j^n\right],$$
for approximating the differential equation
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f$$
for a given function $f(x,t)$ and with fixed $\mu = \Delta t/(\Delta x)^2$, has a truncation error which is $O((\Delta t)^2)$. Here $U_j^n \approx u(x_j,t_n)$, $f_j^n = f(x_j,t_n)$, assuming $u(x,t)$ and $f(x,t)$ are sufficiently smooth.

2. (a) Derive the Lax-Wendroff scheme for solving the advection equation $u_t + au_x = 0$ with constant velocity $a > 0$. Write your result as U_j^{n+1} = linear combination of U_{j-1}^n, U_j^n, U_{j+1}^n, and use $\nu = a\Delta t/\Delta x$. [**Hint**: Do Taylor expansion of $u(x, t+\Delta t)$ in t about (x,t) and replace the time derivatives with space derivatives using the PDE, then use central differences for all space derivatives.]
   (b) Find the amplification factor $\lambda(k)$ in the Fourier analysis of stability.

3. (a) State the CFL condition for a finite difference scheme solving the linear advection equation.
   (b) State the von Neumann Condition for stability of a linear finite difference scheme
$$B_1U^{n+1} = B_0U^n$$
   (c) State the Lax Equivalence Theorem.

4. Calculate the two interpolation functions $\psi_2(x,y)$ and $\psi_6(x,y)$ for the quadratic triangular element shown in the figure. Here nodes 4, 5, and 6 are at the center of the edges. [**Hint: use the area (barycentric) coordinates $L_i(x,y), 1 \le i \le 3$.**]



5. Solve the differential equation $-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 2$ (**note the nonzero right hand side**) with the unit square domain and boundary conditions (left picture below), by a uniform 2×2 mesh and linear rectangular element (right picture below). Show the following steps: (a) weak formulation, (b) element equation (same for all 4 elements), (c) the connectivity matrix, (d) condensed equation

resulted from assembly and imposing boundary conditions (U_5 is the only unknown in the condensed equation) (e) the value of of approximate solution u_h at points node 5.