Ph.D. Comprehensive Examination: Partial Differential Equations

August 29, 1997.

Instructions: This exam consists of two parts, Part I and Part II. Answer any 3 of the 4 questions in each part.

Part I

1. Use the method of characteristics to solve the initial value problem

$$x u_x + u u_y = u \quad , \quad (x,y) \in \mathbb{R} \times \mathbb{R}^+ \tag{1}$$

$$u(x,0) = x \tag{2}$$

for $u = u(x,y)$. For what (x,y) is your solution valid?

2. Let $F(x,y,z,p,q)$ be smooth in its arguments and Γ be a smooth curve in $\mathbb{R}^3$ parametrized by $(X(t), Y(t), U(t))$, $t \in \mathbb{R}$. Let $u(x,y)$ be a solution of

$$F(x,y,u,u_x,u_y) = 0 \tag{3}$$

such that Γ is contained in the graph of u.

i) Define the compatibility conditions associated with u.

ii) State a condition which assures a unique solution to the compatibility conditions.

iii) If $(X(t), Y(t), U(t)) = (t, t, t^2)$ and $F = pq - 1$, what must $u_x(1,1)$ equal?

3. Let $v(x) \in C^2(\bar{\Omega})$ be superharmonic on Ω , that is, $\nabla^2 v \leq 0$ on Ω. Show that

$$v(x) \geq ⨍_{\partial B(x,r)} v(y)\,dS(y) \equiv \frac{1}{S(\partial B)} \int_{\partial B(x,r)} v(y)\,dS(y) \quad \forall B(x,r) \subset \Omega \tag{4}$$

where $B(x,r) = \{x \in \mathbb{R}^n : |x| \leq r\}$ and $S(\partial B)$ is the surface area of $\partial B(x,r)$.

4. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and $u(x,t) \in C^2(\bar{\Omega} \times \mathbb{R}^+)$ be a solution of the initial boundary value problem

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} \quad , \quad (x,t) \in \Omega \times \mathbb{R}^+ \tag{5}$$

$$u(x,0) = f(x) \quad , \quad x \in \Omega \tag{6}$$

$$u_t(x,0) = g(x) \quad , \quad x \in \Omega \tag{7}$$

$$\frac{\partial u}{\partial n} = 0 \quad , \quad x \in \partial\Omega, t > 0 \tag{8}$$

Let

$$E(u,t) = \frac{1}{2} \int\int_{\Omega} \left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2 \right] dx_1 dx_2 \tag{9}$$

denote the energy of u at time t.

i) Prove that the energy $E(u,t)$ is constant.

ii) Formulate a uniqueness theorem for this problem and use the result in (i) to prove it.

Part II

5. Answer all of the following:

a) Define the Sobolev space $H^1(-1,1)$. Is $f(x) = |x|$ in $H^1(-1,1)$? If so, what is $f'(x)$?

b) Define what is meant by the adjoint operator of L on Hilbert space H.

c) Define what is meant by X being imbedded in Y, where X,Y are normed spaces.

d) Define what is meant by a linear elliptic second-order partial differential operator

$$L[u] = -\sum_{i,j=1}^{n} (a_{ij}(x) u_{x_i})_{x_j} + \sum_{i=1}^{n} b_i(x) u_{x_i} + c(x) u \quad , \quad x \in \Omega \subset \mathbb{R}^n \tag{10}$$

on $\Omega \subset \mathbb{R}^n$.

6. Prove that $C_0^\infty(\Omega)$ is a subspace of the Holder space $C^{0,1/2}(\Omega)$. Recall that the norm of $C^{0,\beta}(\Omega)$ is

$$\| f \|_{C^{0,\beta}} \equiv \| f \|_\infty + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\beta}, \qquad 0 < \beta \leq 1 \tag{11}$$

7. State the Fredholm alternative as it applies to the problem

$$L[u] = f \quad , \quad x \in \Omega \tag{12}$$

$$u = 0 \quad , \quad x \in \partial\Omega \tag{13}$$

where L is a symmetric, elliptic second-order partial differential operator and Ω is a smooth bounded domain in $\mathbb{R}^n$. For what value of k does the problem

$$\nabla^2 u = cos^2 x_1 - k \quad , \quad x = (x_1, x_2) \in \Omega = [0,1]^2 \tag{14}$$

$$\frac{\partial u}{\partial n} = 0 \quad , \quad x \in \partial\Omega \tag{15}$$

have a solution. Is the solution unique? Explain.

8. Consider the reaction diffusion system

$$u_t = v - 2u + u_{xx} \tag{16}$$

$$v_t = u - 2v + dv_{xx} \tag{17}$$

$x \in (0,1)$, where Neumann boundary conditions apply at all endpoints for both u, v. Can the spatially homogeneous equilibria of this system undergo a Turing bifurcation? Explain.