PARTIAL DIFFERENTIAL EQUATIONS PRELIMINARY EXAM

Fall 2008

Choose Five problems from the follow eight to work.

1. Find the solution of

$$u^2 u_x + u_y = 0, \quad x \in \mathbb{R}, \quad y > 0$$
$$u(x, 0) = x.$$

Be sure to verify that your solution $u(x, y)$ satisfies the initial condition. When do shocks develop?

2. a) Define what is meant by the α^{th}-weak partial derivative for a function $u \in L^1_{loc}(U)$, for $U \subset \mathbb{R}^n$.

b) Find the weak first derivative of

$$u(x) = \begin{cases} x & \text{for } 0 < x \leq 1 \\ 1 & \text{for } 1 \leq x < 2 \end{cases}$$

on $U = (0, 2)$

c) Show that

$$u(x) = \begin{cases} x & \text{for } 0 < x \leq 1 \\ 2 & \text{for } 1 \leq x < 2 \end{cases}$$

does not have a weak first derivative on $U = (0, 2)$

d) Does the function in part c) have a weak first derivative on $U = (0, 1) \cup (1, 2)$?

3. a) State the Lax-Milgram Theorem.

b) Let

$$Lu = -\Delta u + c(x)u$$

Prove that there exits a constant $\mu > 0$ such that the corresponding bilinear form $B[\ ,\ ]$ satisfies the hypotheses of the Lax-Milgram Theorem provided

$$c(x) \geq -\mu \text{ on } \Omega$$

4. Consider the PDE

$$u_t + F(u)_x = u_{xx} \text{ in } \mathbb{R} \times (0, \infty)$$

where F is a smooth, uniformly convex function.

Show that there is a travelling wave solution $u(x, t) = v(x - ct)$ satisfying $v(-\infty) = u_l$, $v(\infty) = u_r$ if and only if

$$c = \frac{F(u_l) - F(u_r)}{u_l - u_r}$$

5. Let Ω be an open subset of $\mathbb{R}^2$ such that $\Omega \subset B(0,R)$ for some $R>0$ where $B(0,R)$ is the ball of radius R centered at 0.

   a) Let $\Delta u = -F$ in Ω and suppose that $F \leq 0$ in Ω. If in addition $u \in C(\overline{\Omega})$, prove that

   $$\max_{x\in\Omega} u(x) \leq \max_{x\in\partial\Omega} u(x).$$

   b) Consider the nonhomogeneous Dirichlet Problem

   $$\begin{aligned} \Delta u &= -F \quad \text{in } \ \Omega \\ u &= f \quad \text{on } \ \partial\Omega. \end{aligned}$$

   Show that

   $$|u(x,y)| \leq \max_{(x,y)\in\partial\Omega} |f(x,y)| + \frac{1}{4}R^2 \max_{(x,y)\in\Omega} |F(x,y)|.$$

6. a) Use Duhamel's principle to find an explicit solution of

   $$\begin{aligned} &u_{tt}(x,t) = u_{xx}(x,t) + e^x, \quad x\in\mathbb{R}, \quad t>0, \\ &u(x,0)=0, \quad u_t(x,0)=0. \end{aligned}$$

   b) Use d'Alembert's formula to find an explicit solution of

   $$\begin{aligned} &u_{tt}(x,t) = u_{xx}(x,t), \quad x\in\mathbb{R}, \quad t>0, \\ &u(x,0)=\tanh(x), \quad u_t(x,0)=0. \end{aligned}$$

   c) What is the solution of

   $$\begin{aligned} &u_{tt}(x,t) = u_{xx}(x,t) + e^x, \quad x\in\mathbb{R}, \quad t>0, \\ &u(x,0)=\tanh(x), \quad u_t(x,0)=0? \end{aligned}$$

7. Recall that the solution to the initial value heat problem

   $$u_t = u_{xx}, \quad x\in\mathbb{R}, \quad t>0$$

   $$u(x,0) = f(x)$$

   is given by

   $$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4t}} f(y)dy.$$

   a) Prove that the solution depends continuously on the data in the sense that if

   $$|f(x) - \tilde{f}(x)| < \epsilon, \quad -\infty < x < \infty,$$

   then the corresponding solutions satisfy

   $$|u(x,t) - \tilde{u}(x,t)| < \epsilon, \quad -\infty < x < \infty, t>0.$$

   b) Assume that $f(x)$ is continuous and bounded. Show that

   $$\lim_{t\to 0^+} u(x,t) = f(x)$$

8. Consider the following modified heat equation

$$u_t(x,t) = u_{xx}(x,t) - u(x,t), \quad 0 < x < 1, \quad t > 0,$$

$$\begin{aligned} &u(x,0) = f(x), \quad 0 < x < 1, \\ &u(0,t) = 1, \quad u(1,t) = 0, \quad 0 < t < T \ . \end{aligned}$$

(a) Find the steady state solution $u(x,t) = u_{\mathrm{ss}}(x)$.

(b) Use an energy argument on the function

$$w(x,t) = u(x,t) - u_{\mathrm{ss}}(x)$$

to describe the behavior of u as $t \to \infty$.