

PARTIAL DIFFERENTIAL EQUATIONS PRELIMINARY EXAM

Fall 2008

Choose Five problems from the follow eight to work.

1. Find the solution of

$$u^2 u_x + u_y = 0, \quad x \in \mathbb{R}, \quad y > 0$$

$$u(x, 0) = x.$$

Be sure to verify that your solution  $u(x, y)$  satisfies the initial condition. When do shocks develop?

2. a) Define what is meant by the  $\alpha^{th}$ -weak partial derivative for a function  $u \in L^1_{loc}(U)$ , for  $U \subset \mathbb{R}^n$ .

- b) Find the weak first derivative of

$$u(x) = \begin{cases} x & \text{for } 0 < x \leq 1 \\ 1 & \text{for } 1 \leq x < 2 \end{cases}$$

on  $U = (0, 2)$

- c) Show that

$$u(x) = \begin{cases} x & \text{for } 0 < x \leq 1 \\ 2 & \text{for } 1 \leq x < 2 \end{cases}$$

does not have a weak first derivative on  $U = (0, 2)$

- d) Does the function in part c) have a weak first derivative on  $U = (0, 1) \cup (1, 2)$ ?

3. a) State the Lax-Milgram Theorem.

- b) Let

$$Lu = -\Delta u + c(x)u$$

Prove that there exists a constant  $\mu > 0$  such that the corresponding bilinear form  $B[ \cdot, \cdot ]$  satisfies the hypotheses of the Lax-Milgram Theorem provided

$$c(x) \geq -\mu \text{ on } \Omega$$

4. Consider the PDE

$$u_t + F(u)_x = u_{xx} \text{ in } \mathbb{R} \times (0, \infty)$$

where  $F$  is a smooth, uniformly convex function.

Show that there is a travelling wave solution  $u(x, t) = v(x - ct)$  satisfying  $v(-\infty) = u_l$ ,  $v(\infty) = u_r$  if and only if

$$c = \frac{F(u_l) - F(u_r)}{u_l - u_r}$$

5. Let  $\Omega$  be an open subset of  $\mathbb{R}^2$  such that  $\Omega \subset B(0, R)$  for some  $R > 0$  where  $B(0, R)$  is the ball of radius  $R$  centered at 0.

a) Let  $\Delta u = -F$  in  $\Omega$  and suppose that  $F \leq 0$  in  $\Omega$ . If in addition  $u \in C(\overline{\Omega})$ , prove that

$$\max_{x \in \Omega} u(x) \leq \max_{x \in \partial\Omega} u(x).$$

b) Consider the nonhomogeneous Dirichlet Problem

$$\begin{aligned} \Delta u &= -F & \text{in } \Omega \\ u &= f & \text{on } \partial\Omega. \end{aligned}$$

Show that

$$|u(x, y)| \leq \max_{(x, y) \in \partial\Omega} |f(x, y)| + \frac{1}{4} R^2 \max_{(x, y) \in \Omega} |F(x, y)|.$$

6. a) Use Duhamel's principle to find an explicit solution of

$$\begin{aligned} u_{tt}(x, t) &= u_{xx}(x, t) + e^x, & x \in \mathbb{R}, \quad t > 0, \\ u(x, 0) &= 0, \quad u_t(x, 0) = 0. \end{aligned}$$

b) Use d'Alembert's formula to find an explicit solution of

$$\begin{aligned} u_{tt}(x, t) &= u_{xx}(x, t), & x \in \mathbb{R}, \quad t > 0, \\ u(x, 0) &= \tanh(x), \quad u_t(x, 0) = 0. \end{aligned}$$

c) What is the solution of

$$\begin{aligned} u_{tt}(x, t) &= u_{xx}(x, t) + e^x, & x \in \mathbb{R}, \quad t > 0, \\ u(x, 0) &= \tanh(x), \quad u_t(x, 0) = 0? \end{aligned}$$

7. Recall that the solution to the initial value heat problem

$$\begin{aligned} u_t &= u_{xx}, & x \in \mathbb{R}, \quad t > 0 \\ u(x, 0) &= f(x) \end{aligned}$$

is given by

$$u(x, t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} f(y) dy.$$

a) Prove that the solution depends continuously on the data in the sense that if

$$|f(x) - \tilde{f}(x)| < \epsilon, \quad -\infty < x < \infty,$$

then the corresponding solutions satisfy

$$|u(x, t) - \tilde{u}(x, t)| < \epsilon, \quad -\infty < x < \infty, t > 0.$$

b) Assume that  $f(x)$  is continuous and bounded. Show that

$$\lim_{t \rightarrow 0^+} u(x, t) = f(x)$$

8. Consider the following modified heat equation

$$u_t(x, t) = u_{xx}(x, t) - u(x, t), \quad 0 < x < 1, \quad t > 0,$$

$$u(x, 0) = f(x), \quad 0 < x < 1,$$

$$u(0, t) = 1, \quad u(1, t) = 0, \quad 0 < t < T .$$

(a) Find the steady state solution  $u(x, t) = u_{ss}(x)$ .

(b) Use an energy argument on the function

$$w(x, t) = u(x, t) - u_{ss}(x)$$

to describe the behavior of  $u$  as  $t \rightarrow \infty$ .