FALL 2010 PDE PRELIMINARY EXAM

Work six of the following eight problems.

1. Let $D \subset R^d$ be a bounded domain with smooth boundary ∂D. Assume that the complete, orthonormal set of eigenfunctions $\{\phi_n\}$ for the negative Dirichlet Laplacian on D is known, in other words,

$$-\Delta \phi_n = \lambda_n \phi_n, \text{ in } D,\ n = 1, 2, 3, \ldots,$$

with $\phi = 0$ on ∂D, and $\int_D \phi_n^2 dx = 1$.

   a. Given $f \in L^2(D)$, find a closed-form series solution $u(x,t)$ for the initial boundary value problem for the heat equation

$$u_t - \Delta u = 0, \quad \text{for } t > 0, x \in D,$$

$$u(x,t) = 0, \quad x \in \partial D,$$

$$u(x,0) = f \quad x \in D.$$

   b. Show that there exists a constant C such that $|\int_D u(x,t)\phi_n(x)dx| \leq Ce^{-\lambda_n t}$, for all n.

   c. Give an example (the d=1 case will suffice) which shows that the heat equation *does not exhibit* finite propagation speed.

2. For each $n \in \mathbb{N}$, consider the Cauchy problem

$$-\Delta u_n = 0, \quad \text{in } U,$$

$$u_n(x) = \frac{1}{n^2}\sin(nx), \quad \text{on } \{(x,y) : y = 0\},$$

$$\frac{\partial u_n}{\partial y} = \frac{1}{n}, \quad \text{on } \{(x,y) : y = 0\},$$

where $U = \{(x,y) : 0 < y < 1, x \in R\}$. Find a sequence $\{u_n\}$ of solutions to these problems and prove that $\{u_n\}$ does not tend to zero. Explain why this implies that the

Cauchy problem above is not "well posed".

3. a. Let $G \subset R^n$ be an open set. Define what is meant by a distribution on G. Let $f \in L^2(G)$ be a real valued function. Show how f may be identified with a distribution T_f.

   b. let T be a distribution on G and let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multiindex. Define the concept of the α-th distributional derivative of T.

   c. Let

$$r : R^n \to R$$

   be defined by

$$r(x) = \begin{cases} 1 & \text{for } |x| < 1 \\ 0 & \text{for } |x| \geq 1, \end{cases}$$

   where $|\cdot|$ is the Euclidean norm in R^n. Show r may be thought of as a distribution on R^n and compute its distributional gradient ∇r.

4. Let $\Omega \in R^n$ be a bounded domain with smooth boundary $\partial\Omega$ and unit outward normal η. Consider the problem

$$\Delta u - \epsilon u = f, \ \text{ on } \Omega$$

$$\frac{\partial u}{\partial \eta} = 0, \ \text{ on } \partial\Omega.$$

where $f \in L^2(\Omega)$ and $\int_\Omega f = 0$.

   a. Prove that for each real $\epsilon > 0$, there exists a unique weak solution $u_\epsilon \in H^1(\Omega)$.

   b. In the case of $\epsilon = 0$, find an additional condition on u which guarantees the existence of a unique solution u_0, such that the solutions u_ϵ from part (a) converge to u_0 as $\epsilon \to 0^+$. Prove existence and uniqueness of u_0 (with the extra condition), and prove that $u_\epsilon \to u_0$ in $H^1(\Omega)$ as $\epsilon \to 0^+$.

5. Consider the first order equation

$$u_t + \tan(u)u_x = 0, \; x \in R, \;\; t \geq 0,$$

$$u(x,0) = \phi(x), \; x \in R.$$

a. Find a solution to this problem where $\phi(x) = \arctan(x)$.

b. For

$$\phi(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ \frac{\pi}{4} & \text{for } x > 0, \end{cases}$$

find two weak solutions to the above problem. Prove that one of them is a weak solution.

6. Let $\Omega \subset R^n$ be a bounded smooth domain, $0 < T < \infty$ be a constant, $c \in C^\infty(\bar{\Omega})$ such that $c(x) \leq 0$ for all $x \in \Omega$ and $\psi \in C^\infty(\bar{\Omega} \times [0,T])$. Let $u \in C^{2,1}(\bar{\Omega} \times [0,T])$ be a solution to

$$u_t = \Delta u + c(x)u, \quad \text{if } (x,t) \in \Omega \times [0,T],$$

$$u(x,t) = \psi(x,t), \quad \text{if } (x,t) \in \Omega \times \{0\} \quad \text{ or if } (x,t) \in \partial\Omega \times [0,T].$$

a. Prove the Maximum Principle: If $\psi(x,t) \geq 0$ then the solution satisfies $u(x,t) \geq 0$.

b. Show that solutions to the above initial-boundary value problem are unique.

7. Use Duhamel's principle to find an explicit solution of

$$u_{tt}(x,t) = u_{xx}(x,t) + \sin(x-t), \quad x \in \mathbb{R}, \quad t > 0,$$
$$u(x,0) = 0, \quad u_t(x,0) = 0.$$

8. Consider the Dirichlet problem

$$\Delta u = f, \quad \text{in } \Omega,$$
$$u(x) = 0, \quad \text{on } \partial\Omega,$$

where $\Omega \subset R^n$ is a bounded domain and $f \in L^2(\Omega)$.

a. Define the concept of a weak $H_0^1(\Omega)$ solution of the above problem.

b. Use the Poincaré inequality:

$$\int_\Omega u^2 dx \le c \int_\Omega |\nabla u|^2 dx \quad \text{for all } u \in H_0^1(\Omega).$$

to prove the existence of a weak solution in $H_0^1(\Omega)$