WINTER 2011 PDE PRELIMINARY EXAM

Choose Five of the Following Eight Problems to Work

1. Consider the Cauchy Problem for the heat equation,

$$u_t = u_{xx}, \quad x \in \mathbb{R},\ t > 0, \quad u(x,0) = h(x)\ x \in R.$$

   a. Use the Fourier Transform to derive the solution to the above initial value problem.

   b. Consider the non-homogeneous heat equation

$$u_t = u_{xx} + f(x,t), \quad x \in \mathbb{R},\ t > 0, u(x,0) = 0, \quad x \in \mathbb{R}$$

   Use the result above and Duhamel's principle to find the following formula for the solution of this problem:

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi(t-\tau)}} e^{\frac{-(x-\xi)^2}{4(t-\tau)}} f(\xi,\tau)\, d\xi d\tau.$$

2. Formulate and prove the maximum principle for the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

   in the strip $\Omega = \{(x,t) : 0 < x < l, 0 < t < T\}$.

3. The quasilinear Cauchy problem

$$xu_x + yuu_y + xy = 0, \quad u = 5 \text{ when } xy = 1$$

   has the solution

$$u(x,y) = -1 + \sqrt{38 - 2xy}.$$

   a. Derive this solution by the method of characteristics. (Hint: Along a characteristic $x(s), y(s), u(s)$, find dz/ds where $z = xy$.)

b. Verify the $u(x, y)$ is a solution in the domain defined by $xy < 19$.

4. Consider the first order equation

$$u_t + uu_x = 0, \ x \in R, \ \ t \geq 0,$$

$$u(x, 0) = h(x), \ x \in R.$$

a. Define what it means to be a weak solution of the above problem.

b. For $u_0 > 0$ and

$$h(x) = \begin{cases} u_0 & \text{for } x \leq 0 \\ u_0(1-x) & \text{for } 0 < x < 1, \\ 0, & \text{for } x \geq 1, \end{cases}$$

Show that a shock develops at a finite time and describe the global weak solution.

5. Let

$$r(x) = \begin{cases} x^2 & \text{for } x \geq 0 \\ 0 & \text{for } x \leq 0. \end{cases}$$

Compute **all** distributional derivatives of r: $D(r)$, $D^{(2)}(r), \ldots$

6. Let G be an open subset of R^n.

a. Let $f \in L^2(G)$. Show how f defines a continuous linear functional on $H^1(G)$.

b. State the Lax-Milgram theorem.

c. Show that the Lax-Milgram theorem may be applied to the bi-linear mapping

$$B[u, v] = \sum_{i=1}^{n} \int_G \partial_i u \partial_i v dx + \int_G uv dx$$

and f as in part a) above.

d. Assume that G is a bounded open set with C^∞ boundary and let u be the Lax-Milgram solution just determined. What partial differential equation (in the sense of distributions) does u solve?

7. Let $D \subset R^d$ be a bounded domain with smooth boundary ∂D. Assume that the complete, orthonormal set of eigenfunctions $\{\phi_n\}$ for the negative Dirichlet Laplacian on D is known, in other words,

$$-\Delta\phi_n = \lambda_n\phi_n, \text{ in } D,\ n = 1, 2, 3, \ldots,$$

with $\phi = 0$ on ∂D, and $\int_D \phi_n^2 dx = 1$ and each $\lambda_n > 0$.

a. Given $f \in L^2(D)$, find a closed-form series solution $u(x,t)$ for the initial boundary value problem for the wave equation

$$u_{tt} - \Delta u = 0, \quad \text{for } t > 0, x \in D,$$

$$u(x,t) = 0, \quad x \in \partial D,$$

$$u(x,0) = f \quad x \in D.$$

$$u_t(x,0) = 0 \quad x \in D.$$

b. Show that

$$E(t) = \frac{1}{2}\int_D u_t^2 + |\nabla u|^2 dx$$

is constant.

8. Let

$$g(x) = \begin{cases} 1 - |x| & \text{when} \quad |x| \le 1; \\ 0 & \text{when} \quad |x| > 1 \end{cases}$$

Sketch profiles of the solutions of two Cauchy problems (1), (2) at time $t = 1$.a

$$u_{tt} = u_{xx}, \qquad u|_{t=0} = g(x), \qquad u_t|_{t=0} = 0,\ x \in R. \tag{1}$$

$$u_t = u_{xx}, \qquad u|_{t=0} = g(x),\ x \in R \tag{2}$$