Real Analysis Comprehensive Exam

Fall 2005

1. Let $\mathbb{X}$ be σ-algebra of subsets of the set X and let $f : X \to \mathbb{R}$ be a non-negative measurable function. Prove that there are measurable simple functions $\phi_n : X \to \mathbb{R}$ with the properties:

   i) $\lim_{n\to\infty} \phi_n(x) = f(x)$ for all $x \in X$

   ii) $\phi_n(x) \le \phi_{n+1}(x)$ for all $x \in X$ and $n = 1, 2, \cdots$.

2. Let λ denote Lebesgue measure on $\mathbb{R}$, let B denote the σ-algebra of Borel subsets of $\mathbb{R}$, and let $f : \mathbb{R} \to \mathbb{R}$ be continuous and non-negative. Let μ be the measure defined by $\mu(A) = \int_A f d\lambda$, $A \in B$

   i. Prove that μ is σ-finite.

   ii. Prove that if f is (Lebesgue) integrable then: for each $\epsilon > 0$ there is a $\delta > 0$ so that if $A \in B$ and $\lambda(A) < \delta$ then $\mu(A) < \epsilon$.

   iii. Suppose that $g : \mathbb{R} \to \mathbb{R}$ is non-negative and Borel measurable. Prove that $\int g d\mu = \int g f d\lambda$.

   iv. Suppose that $\int f^3 d\lambda < \infty$. For what number $r \ge 1$ does $g \in L_r(\lambda) \Rightarrow g \in L_1(\mu)$? (explain)

3. Justify: $\lim_{t\to 0} \int_0^\infty \sum_{n=1}^\infty \frac{2\sin(xt)}{nt} e^{-(nx)^2} dx = \sum_{n=1}^\infty \frac{1}{n^2}$