Real Analysis

Ph. D. Comprehensive Exam

August 21, 2006

On this exam, all measures are Lebesgue measure.

Do all parts of problem 1 and work two of the last three.

1. Prove or provide a counterexample (with justification).
   (a) If $A \subset \mathbb{R}$ is countable then the measure of A is zero.
   (b) If $A \subset \mathbb{R}$ has measure zero then A is countable.
   (c) If $f : \mathbb{R} \to \mathbb{R}$ is integrable then f^2 is integrable.
   (d) If $f : [0,1] \to \mathbb{R}$ is measurable and f^2 is integrable, then f is integrable.
   (e) If $f_n : \mathbb{R} \to \mathbb{R}$ is a sequence of measurable functions and $f_n \to f$ a.e., then $f_n \to f$ in measure.
2. Prove that if $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(x) - f(y)| \le 6|x - y|$ for all $x, y \in \mathbb{R}$ and $A \subset \mathbb{R}$ has measure zero, then $f(A)$ also has measure zero.
3. (a) Give an example of a sequence of L^1 functions g_n with $\lim_{n\to\infty} \int_0^1 |g_n| \, d\mu = 0$ but $g_n \not\to 0$ a.e.
   (b) Prove that if $f_n : [0,1] \to \mathbb{R}$ is a sequence of measurable functions with $\int_0^1 |f_n| \, d\mu \le \frac{1}{n^2}$ for all n, then $f_n \to 0$ a.e.
4. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is L^1, that $g : \mathbb{R} \to \mathbb{R}$ is continuous and periodic with period 1, and that $\int_0^1 g \, d\mu = 0$. Let $g_n(x) := g(nx)$. Find $\lim_{n\to\infty} \int_{-\infty}^{\infty} f g_n \, d\mu$. (Hint: first consider step functions f.)