Real Analysis Comprehensive Exam
August 22, 2007

Do all problems. On this exam, λ denotes Lebesgue measure.

1. True or false? Justify your answers.

(a) If $f : \mathbb{R} \to \mathbb{R}$ is Borel measurable, then $\mu(A) = \int_A f^2 \, d\lambda$ is a Borel measure.

(b) If $f : \mathbb{R} \to \mathbb{R}$ is non-negative and Borel measurable, then $\mu(A) = \left(\int_A f \, d\lambda\right)^2$ is a Borel measure.

(c) If $f_n \to f$ in $L_2([0,1])$, then this convergence also holds in $L_1([0,1])$.

(d) If $f_n \to f$ in $L_2([0,1])$, then this convergence also holds in $L_\infty([0,1])$.

(e) There exists a Borel charge μ on $\mathbb{R}$ with $\mu((a,b]) = \sin b - \sin a$ for all $a, b \in \mathbb{R}$, $a < b$.

2. Find (with justification)

$$\lim_{n\to\infty} \int_0^1 \frac{\cos \frac{1}{nx}}{\sqrt{x}} dx.$$

3. Let $f, g : [0,1] \to [0,\infty)$ be Borel measurable functions with $f(x)g(x) \geq 1$ for (Lebesgue) almost every $x \in [0,1]$. Show that

$$\int_{[0,1]} f \, d\lambda \cdot \int_{[0,1]} g \, d\lambda \geq 1.$$

(Hint: Cauchy-Schwarz inequality.)

4. Let $f, g : \mathbb{R} \to [0,\infty)$ be Borel measurable functions, and let $\mu(A) = \displaystyle\int_A f \, d\lambda$, $\nu(B) = \displaystyle\int_B g \, d\lambda$ for Borel sets $A, B \subseteq \mathbb{R}$.

(a) Show that μ and ν are σ-finite Borel measures absolutely continuous with respect to Lebesgue measure.

(b) Let π be the product measure of μ and ν on $\mathbb{R}^2$. Find the Radon-Nikodym derivative of π with respect to two-dimensional Lebesgue measure λ^2. Justify your answer.