Real Analysis Comprehensive Exam
August 25, 2008

Do all problems. On this exam, λ denotes Lebesgue measure.

1. True or false? Justify your answers. (In parts (c) and (d) the measure is Lebesgue measure on the Borel σ-algebra.)

(a) If $f : \mathbb{R} \to \mathbb{R}$ is non-decreasing, then f is Borel-measurable.

(b) If μ_1 and μ_2 are measures on a measurable space $(X, \mathbb{X})$, then $\nu(A) = \max(\mu_1(A), \mu_2(A))$ is a measure on $(X, \mathbb{X})$.

(c) If $f \in L_3([0,1])$, then $f \in L_2([0,1])$.

(d) If $f_n : [0,1] \to \mathbb{R}$ is a sequence of measurable functions with $f_n \to 0$ a.e., then $f_n \to 0$ in measure.

2. Find (with justification)

$$\lim_{n\to\infty} n \int_0^\infty e^{-x^2} \sin \frac{x}{n}\, dx.$$

3. Let $\sum_{n=1}^{\infty} a_n$ be a convergent series with non-negative terms. Show that $\sum_{n=1}^{\infty} \frac{a_n^t}{n}$ converges for every $t > 0$. (Hint: For $t \geq 1$ this is just a calculus exercise. For $0 < t < 1$ use Hölder's inequality for the counting measure.)

4. Let μ be a Borel measure on $\mathbb{R}$ (i.e., a measure on the Borel σ-algebra) with $\mu([a,b)) \leq b - a$ for all $a < b$. Show that $\mu \ll \lambda$ with $\frac{d\mu}{d\lambda} \leq 1$ λ-a.e.