REAL ANALYSIS QUALIFYING EXAMINATION QUESTIONS

1. True or False? If True, prove; if False, give a counterexample.

   a. If A is a subset of R and $m^*(A) = 0$, then for any $a, b \in A$, $a < b$, there is an $x \in R - A$ such that $a < x < b$. (Here m^* is Lebesgue outer measure.)

   b. If A is a subset of $[0, 1]$ and $m^*(A) = 1$, then A contains an interval of positive length. (Here m^* is Lebesgue outer measure.)

   c. If $E \subset [0, 1]$ satisfies $m^*(E) + m^*([0, 1] - E) = 1$ then E is measurable. (Here m^* is Lebesgue outer measure.)

   d. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots$ be a sequence in $\mathcal{M}$. Then $\mu(\cap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$.

   Do any **three** of the following **four** questions:

2. Let (X, μ) be a measure space and let f be a measurable function on X.

   (a) Assume f is bounded and $f \in L^1(X)$. Prove that $f \in L^2(X)$.

   (b) Assume $\mu(X) < \infty$, and $f \in L^2(X)$. Prove that $f \in L^1(X)$.

3. Let $f : R \to R$ be a positive function. Define

$$X = \{(x, y) \in R^2 | 0 \le y \le f(x)\}$$

   Assume that X is Lebesgue measurable and its measure is finite. Show that the function f is integrable, and the measure of X is equal to

$$\int_R f(x)dx.$$

4. Suppose that E is a Lebesgue measureable subset of R and that the Lebesgue measure of E is finite. Suppose also that for each $n \in \mathbb{N}$, $f_n : E \to R$ is a bounded , measurable, non-negative function on E and that $\lim_{n\to\infty} f_n(x)$ exits in R. Let $f(x) = \lim_{n\to\infty} f_n(x)$. Show by counterexample that without further hypotheses, the following assertion is false:

$$\int_E f(x)dx = \lim_{n\to\infty} \int_E f_n(x)dx$$

   What additional hypothesis are required to make the above assertion true (but still non trivial)?

5. If E is a measurble set in R^d and $x \in R^d$, we say that x is a point of Lebesgue denisty of E if

$$\lim_{r\to 0^+} \frac{m(B_r(x) \cap E)}{m(B_r(x))} = 1$$

   where $B_r(x)$ is the ball in R^d centered at x with radius r and m is Lebesgue measure. Suppose that 0 is a point of Lebesgue density of A and of B, both measureable subsets of R^d. Show that 0 is also point of Lebesgue density of $A \cap B$.