Ph.D. Comprehensive Exam: Real Analysis
August 2010

1. True or False? If True, prove; if False, give a counterexample.

   a. Suppose that A and B are subsets of R such that the Lebesgue outer measure of A is zero, i.e., $m^*(A) = 0$ then $m^*(A \cup B) = m^*(B)$.

   b. If A is a subset of $[0, 1]$ and $m^*(A) = 1$, then A contains an interval of positive length.

   c. If $f_n \to 0$ in $L_p([0, 1])$ for some $p \in [1, \infty)$ then $f_n \to 0$ a.e.

   d. If m is the Lebesgue measure on R^2 and if $f : R \to R$ is Borel-measurable then

$$m\{(x, f(x)) : x \in R\} = 0.$$

2. Evaluate

$$\lim_{n\to\infty} \int_0^\infty \frac{e^{-x(3+n^{-2}\sin(x))}}{1 + n^{-2}x} dx$$

   justifying your answer.

3. Assume that f is non-negative and Lebesgue integrable on R^d.

   Show that for every $\epsilon > 0$ there exists a $\delta > 0$ such that if A is a measurable subset of R^d such that $\mu(A) \leq \delta$ then

$$\int_A f d\mu \leq \epsilon.$$

4. Let μ, ν, λ be σ-finite positive measures on $(X, \mathcal{M})$.

   a. Show that $\mu \ll \mu + \lambda$.

   b. Show that if $\nu \ll \mu$ and $\lambda \ll \mu$ then $\nu + \lambda \ll \mu$ and

$$\frac{d(\nu + \lambda)}{d\mu} = \frac{d\nu}{d\mu} + \frac{d\lambda}{d\mu} \qquad \mu - \text{a.e.}$$

   c. Show that if $\lambda \ll \nu \ll \mu$ then $\lambda \ll \mu$ and

$$\frac{d\lambda}{d\mu} = \frac{d\lambda}{d\nu}\frac{d\nu}{d\mu} \qquad \mu - \text{a.e.}$$