

Real Analysis Ph. D. Comprehensive Exam

August 2011

Do all parts of problem 1, and work 3 of the other 4 problems.

If not explicitly specified, the measure space is \mathbb{R} with the Borel σ -algebra \mathbf{B} and Lebesgue measure λ .

1. True or false? Justify your answers.

(a) If (f_n) is a sequence of measurable non-negative functions on a measure space (X, \mathbf{X}, μ) , then $\int \sum_{n=1}^{\infty} f_n d\mu = \sum_{n=1}^{\infty} \int f_n d\mu$.

(b) If μ_1 and μ_2 are σ -finite measures with $\mu_1 \ll \mu_2$, then there exists a μ_2 -integrable function f with $\mu_1(A) = \int_A f d\mu_2$ for all measurable sets A .

(c) For every $f \in L_2$ there exists a sequence of functions $\phi_n \in L_2$, vanishing outside $[-n, n]$, such that $\phi_n \rightarrow f$ in L_2 .

(d) For every $f \in L_\infty$ there exists a sequence of functions $\phi_n \in L_\infty$, vanishing outside $[-n, n]$, such that $\phi_n \rightarrow f$ in L_∞ .

2. Let (X, \mathbf{X}, μ) be a measure space, and let $(E_k)_{k=1}^{\infty}$ be a sequence of measurable sets such that $\mu(E_k) \geq 2011$ for all $k \in \mathbb{N}$. Let E be the set of points in X which belong to E_k for infinitely many indices k .

(a) Show that E is measurable.

(b) Under the additional assumption that $\mu(X) < \infty$, show that $\mu(E) \geq 2011$.

(c) Give an example to show that the additional assumption in (b) is necessary.

3. Let $f_n : \mathbb{R} \rightarrow \mathbb{R}$ be a sequence of Borel-measurable functions converging to 0 uniformly, and satisfying $\int_{\mathbb{R}} |f_n| d\lambda \leq 1$ for all n .

(a) Show that $f_n \rightarrow 0$ in L_p for all $p > 1$.

(b) Give an example to show that the assumptions do not imply $f_n \rightarrow 0$ in L_1 .

4. For $\alpha \geq 0$ let $F(\alpha) = \int_0^1 \frac{x^\alpha - 1}{\ln x} dx$.

(a) Show that F is differentiable with $F'(\alpha) = \frac{1}{\alpha + 1}$.

(b) Use this result to calculate $\int_0^1 \frac{x - 1}{\ln x} dx$.

5. Let f be a non-negative measurable function on a σ -finite measure space (X, \mathbf{X}, μ) . Show that $\int_X f d\mu = \int_0^\infty \mu(F_t) dt$, with $F_t = \{x \in X : f(x) > t\}$.