

Real Analysis Ph. D. Comprehensive Exam

August 2012

Solve 4 of the following 5 problems.

In problems 2–5, the measure is Lebesgue measure λ on the Borel σ -algebra \mathbf{B} . Integrals of the form $\int_a^b f(x) dx$ are to be understood as Lebesgue integrals.

1. Let (X, \mathbf{X}, μ) be a measure space, let E_n be a sequence of measurable sets with $\mu(X \setminus E_n) \rightarrow 0$, and let G be the set of $x \in X$ belonging only to finitely many sets E_n . Show that G is measurable and that $\mu(G) = 0$.

2. Let $E \subset [0, \infty)$ be a measurable set. Show that for any number $c \in [0, \lambda(E))$ there exists $a \geq 0$ with $\lambda([0, a] \cap E) = c$.

3. Let $f_n, f : \mathbb{R} \rightarrow [0, \infty)$ be integrable functions with $f_n \rightarrow f$ a.e., and $\int f_n d\lambda \rightarrow \int f d\lambda$. Show that $\int |f_n - f| d\lambda \rightarrow 0$. (Hint: Apply the dominated convergence theorem to the sequence of functions $g_n = \min(f, f_n)$.)

4. With justification, find

$$\lim_{n \rightarrow \infty} \int_0^1 \frac{dx}{x^{\frac{1}{n}} \left(1 + \frac{x}{n}\right)^n}.$$

5. Let $f : [0, a] \times [0, b] \rightarrow [0, +\infty)$ be a measurable function such that

$$\int_0^a f(x, y) dx \geq 1$$

for almost all $y \in [0, b]$.

(a) Show that

$$\int_0^a \int_0^b f(x, y)^2 dy dx \geq \frac{b}{a}.$$

(b) What can you conclude about f if equality holds in (a)?

(Hint: Cauchy-Schwarz and Fubini-Tonelli.)