

# Real Analysis Ph. D. Comprehensive Exam

January 2012

Do all parts of problem 1, and work 3 of the other 4 problems.

If not explicitly specified, the measure space is  $\mathbb{R}$   
with the Borel  $\sigma$ -algebra  $\mathbf{B}$  and Lebesgue measure  $\lambda$ .

All functions are assumed to be measurable.

1. True or false? Justify your answers.

(a) The Lebesgue measure of an open dense subset of  $\mathbb{R}$  is infinite.

(b) If  $p, q > 1$ ,  $1/p + 1/q = 1$ ,  $f_n \rightarrow f$  in  $L_p$ , and  $g_n \rightarrow g$  in  $L_q$ , then  $f_n g_n \rightarrow fg$  in  $L_1$ .

(c) If  $1 \leq p < \infty$ ,  $f_n \geq 1$ , and  $f_n \rightarrow f$  in  $L_p$ , then  $1/f_n \rightarrow 1/f$  in  $L_p$ .

(d) If  $1 \leq p < \infty$ ,  $|f_n| \leq 1$ ,  $f_n \rightarrow f$  in measure, and  $g_n \rightarrow g$  in  $L_p$ , then  $f_n g_n \rightarrow fg$  in  $L_p$ .

2. Let  $\mu$  be a Borel charge (i.e., a signed Borel measure) on  $[0, 1]$  with  $\int_{[0,1]} f d\mu \geq 0$  for every continuous  $f \geq 0$ . Show that  $\mu$  is a measure. (For partial credit replace the word “continuous” by “measurable”.)

3. Let  $f$  be an integrable function, and let  $(A_n)_{n=1}^{\infty}$  be a sequence of measurable sets such that  $\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k = \emptyset$ , i.e., such that every  $x$  is contained in  $A_n$  only for finitely many indices  $n$ . Show that  $\int_{A_n} f(x) dx \rightarrow 0$  as  $n \rightarrow \infty$ .

4. Let  $f(x) = \int_0^{\infty} \frac{1 - e^{-xt^2}}{t^2} dt$  for  $x > 0$ . Show that  $f$  is differentiable and find an explicit formula for  $f'$ . Use this to find an explicit formula for  $f$ . (You may use the fact that  $\int_0^{\infty} e^{-s^2} ds = \frac{\sqrt{\pi}}{2}$  without proof.)

5. Let  $f$  be a Borel-measurable real-valued function on  $[0, 1]$  such that  $g(x, y) = |f(x) - f(y)|$  is integrable on  $[0, 1] \times [0, 1]$  with respect to two-dimensional Lebesgue measure. Show that  $f$  is integrable on  $[0, 1]$ .