

Ph.D. Exam in Topology
August 18, 1993

Instructions: For a passing grade you must work at least five of the following problems.

1. Suppose that $f : X \rightarrow Y$ is a continuous and 1-1 map from X onto Y . If X is compact and Y is Hausdorff, prove that f is a homeomorphism.

2. Use the homotopy or homology structure of the circle to prove that every continuous map on the closed disc $f : \mathbb{D} \rightarrow \mathbb{D}$ admits a point $p \in \mathbb{D}$ for which $f(p) = p$.

3. Let $f : (0, 1] \rightarrow [0, 1]$ denote a continuous function with the property that

$$f\left(\frac{1}{2n}\right) = 0 \quad \text{and} \quad f\left(\frac{1}{2n-1}\right) = 1,$$

for all $n = 1, 2, 3, \dots$. Prove that the set

$$X = \text{Graph}(f) \cup (\{0\} \times [0, 1])$$

is connected but not path connected.

4. Let X be path connected. Prove that the following are equivalent:

- a) X is simply connected.
- b) Every map of the unit circle \mathbb{S}^1 into X extends to a map from the closed unit disc into X .
- c) If f and g are paths in X such that $f(0) = g(0)$ and $f(1) = g(1)$ then $f \simeq g$ (*endpoint homotopic*).

5. Suppose that A is a subset of the topological space X . Suppose that $x \in A$ and y lies in the set complement $X \setminus A$ of A . If φ denotes a path in X joining $x = \varphi(0)$ to $y = \varphi(1)$, show that there exists a “time” t_* at which $\varphi(t_*)$ lies in the boundary ∂A .

6. Prove that every continuous open mapping from \mathbb{R} to \mathbb{R} is a homeomorphism onto its image. Recall that an open mapping f from X to Y has the property that if V is an open set in X , then $f(V)$ is an open set in Y .

7. In this problem you may use the fact that the homotopy group of the figure eight, denoted “8”, in the plane is the *free group* $\mathcal{F}_2(\alpha, \beta)$ on two symbols. Let $f : 8 \rightarrow \mathbb{S}$ be the map which folds the figure eight at the intersection point to make a circle. Determine exactly the induced homotopy map $f_* : \pi_1(8) \rightarrow \pi_1(\mathbb{S}^1)$. Identify the kernel of this homomorphism.