

Ph.D. Exam in Topology
August 19, 1994

Instructions: Attempt all of the following problems. Partial credit will be assigned to work which leads to a solution. If you believe there is an error in a problem please bring it to the attention of one of the examiners.

1. Suppose X is a topological space that is connected and locally path connected. Prove that X is path connected.
2. Let (X, d) be a compact metric space. Suppose that $f : X \rightarrow X$ is a continuous map. Then show that either $f(x) = x$ for some $x \in X$ or there exists a number $\epsilon > 0$ such that $d(f(x), x) > \epsilon$ for all $x \in X$.
3. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ denote a continuous function which is periodic; that is, there exists a number $T > 0$ such that $f(x + T) = f(x)$ for all $x \in \mathbb{R}$. Show that f is uniformly continuous.
4. Suppose \mathbb{D} denotes the closed unit disc in complex plane \mathbb{C} . Prove or provide a counterexample: if $f : \mathbb{D} \rightarrow \mathbb{C}$ is continuous with $f(z) = z^2$ for all $z \in \partial\mathbb{D}$, then $f(\mathbb{D}) \supset \mathbb{D}$. The notation ∂A means the boundary of the set A .
5. Let $f, g : [a, b] \rightarrow \mathbb{R}$ be continuous. Let S_f denote the union of the graph of f and the graph of $-f$ for $a \leq x \leq b$. Create the set S_g in the same fashion. Let Z_f and Z_g denote respectively the zero sets of f and g . Prove that S_f and S_g are homeomorphic subspaces of the plane if and only if there is a bijection from Z_f onto Z_g that extends to a homeomorphism of $[a, b]$ onto $[a, b]$.