Topology PhD Comprehensive Exam

August 27, 2003

1. Suppose that X is a compact Hausdorff space and that A and B are disjoint closed subsets of X. Prove that there are open sets U and V with $A \subset U$, $B \subset V$ and $\bar{U} \cap \bar{V} = \emptyset$.

2. Suppose that X is a locally connected space and that $f : X \to \mathbb{R}$ is continuous with the property:

$$\text{int}(f^{-1}(\{x\})) = \emptyset \quad \text{for each} \quad x \in \mathbb{R}.$$

   a) Prove that if A is a dense subset of $\mathbb{R}$ then $f^{-1}(A)$ is dense in X.

   b) Give a counter example to a) if X is not assumed locally connected.

3. Let K be the middle thirds Cantor set in $[0, 1]$. Is there a real number x with the property that $x + r \notin K$ for every rational number r?

4. In each case below, prove that there is no retraction $r : X \to A$. (Recall that if $A \subset X$, a retraction $r : X \to A$ is a continuous map with the property that $r(a) = a$ for each $a \in A$.)

   a) X is a 2-simplex, A is its boundary.

   b) $X = S^1 \times [0, 1]$, $A = S^1 \times \{0, 1\}$, $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

   c) $X = S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$.
   $A = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1,\ z = 0\}$.

   d) $X = S^1 \times S^1$, $A = (S^1 \times \{p\}) \cup (\{p\} \times S^1)$, some $p \in S^1$.

5. Let K be the 2-complex pictured (so |K| is a Mobious band) and let L be the subcomplex consisting of the edges ae, ef, fb, bc, cd, da and their vertices (so |L| is the boundary of |K|).

   a) Use an exact sequence to compute the homology groups of the pair (K, L).

   b) Is there a retraction $r : |\mathrm{K}| \to |\mathrm{L}|$?

   c) Let K′ be the 2-complex

      (Note that $|\mathrm{K}'| \cap |\mathrm{K}| = |\mathrm{L}|$.) Use an exact sequence to compute the homology groups of $|\mathrm{K}| \cup |\mathrm{K}'|$.