Ph.D. Comprehensive Exam - Topology

August 2012

In the following, $\mathbb{R}$ is the set of real numbers. Every answer to a question in this exam needs a proof.

(1) Let τ be the collection of subsets U of $\mathbb{R}$ such that $U \supset (0,1)$, together with the empty set.

(a): Show that τ is a topology on $\mathbb{R}$.

(b): Find the closure of the interval $(0,1)$ in $(\mathbb{R}, \tau)$.

(c): Find the interior of the interval $(0,1)$ in $(\mathbb{R}, \tau)$.

(d): Let $(\mathbb{R}, u)$ denote $\mathbb{R}$ with its usual topology. Is the function $f : (\mathbb{R}, u) \to (\mathbb{R}, \tau)$ defined by $f(x) = x$ continuous? Is the function $g : (\mathbb{R}, \tau) \to (\mathbb{R}, u)$, defined by $g(x) = x$ continuous?

(2) Let $f : X \to Y$ be a continuous map of Hausdorff spaces. Let $B_1 \supset B_2 \supset \ldots \supset B_n \supset \ldots$ be a decreasing sequence of compact subsets of X. Prove that

$$f\left(\bigcap_{i=1}^{\infty} B_i\right) = \bigcap_{i=1}^{\infty} f(B_i).$$

(3) (a) Show that any continuous $f : S^2 \to S^1 \times S^1$ must be null-homotopic (i.e., homotopic to a constant map).

(b) Show that there is a continuous map $g : S^1 \times S^1 \to S^2$ that is not null-homotopic.

(4) View the Klein bottle K as a union of two Mobius bands M_1 and M_2 identified along their boundaries. Compute $H_*(K)$.