The Introduction to the Analysis of Balanced Lattice Designs Using SAS

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APPROVAL

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Introduction

The lattice designs are one class of incomplete block designs. There are several types of lattice designs including balanced lattices and partially balanced lattices. In this paper, the balanced lattices are considered. In a lattice design, the number of treatments must be an exact square. The number of blocks and the number of units in each block is the square root of the number of treatments. These incomplete blocks are combined into groups that form separate, complete replications of all treatments.

The main advantage of balanced lattices is that a large number of treatments may be compared within relatively small blocks. Another advantage of balanced lattice designs is that each pair of treatments is compared with the same degree of precision because each treatment occurs together in the same block with every other treatment an equal number of times (usually once). Hence, to obtain a balanced lattice, some restrictions on the number of treatments and the number of blocks in the design are required. Consequently, balanced lattices are not available for 36, 100, and 144 treatments. The disadvantages of the design are the limitations for the number of allowable treatments, block sizes, and replication. The analysis also becomes more complex and the designs are more difficult to construct as the number of treatments increase.

The lattice design is most commonly used in agricultural field experiments. There is sufficient flexibility in the design to make its application simpler than other incomplete block designs. For example, in agricultural research, it is often difficult to find a sufficient number of uniform plots to form blocks large enough to contain complete replication of all treatments. Thus, the effect of blocking is lost unless block size can be made smaller than the number of treatments to be tested.

A brief history

Designs such as factorial designs and randomized complete block designs were unsuitable for experiments in which a large number of treatments are used. Consequently, F. Yates developed the group of incomplete block designs known as quasi-factorials or lattices in 1936. In 1939, Yates wrote the article " The Recovery of Inter-block Information in Variety Trials Arranged in Three-dimensional Lattices". In 1940, F. Yates wrote the article " Lattice Squares" that was one of a series of papers describing new methods of analysis for lattice and incomplete block designs. He mentioned that the information contained in the inter-row and inter-column comparisons was recovered when the lattice square was used. Moreover, the lattice square was useful when large numbers of varieties were compared, especially in the case that latin squares were found to be effective in reducing the variability of the experiment material. In the same year, G. M. Cox, R. C. Eckhardt, and W. G. Cochran constructed the lattice and triple lattice experiments used to analyze yield in corn. The two experiments consisted of testing the yield of 81 double-crosses of corn. They stated that the recovery of inter-block information and the reduction of block size from 81 to 9 plots per block resulted in a remarkable increase in precision when compared to the randomized complete block designs. In addition, these designs were especially desirable when little was known about the variability of the experimental field. In 1941, W. G. Cochran studied the accuracy of lattice and lattice square experiments on corn and found that on the average three replications of a triple lattice were somewhat more accurate than five replications of the randomized complete block and the relative precision varied from 114 to 365 percent. Also, in the lattice square design, the relative precision ranged from 98 to 462 percent with an average saving of one replication in six for tests of 25 varieties to one replication in three for tests with 121 varieties. Furthermore, the standard error increased slightly with increasing numbers of varieties in the test which indicated the value of these designs in providing accurate comparisons for tests with many varieties. In 1943, I. J. Johnson, and H. C. Murphy studied the lattice and lattice square designs with oat uniformity data and in variety trials. They found that an arrangement of the blocks in lattice designs reduced the error variance of the randomized complete block, and the average precision for the lattice and lattice square designs in comparison with randomized complete blocks ranged from 155 to 224 percent. In 1945, S. W. Boyce determined the efficiency of lattice designs on New Zealand wheat trials. The results from twenty-one lattice trials showed an increase of efficiency over the randomized complete block design ranging from 0 to 152 percent, and the mean increase was 18 percent. In 1948, R. E. Comstock, W. J. Peterson, and H. A. Stewart used a balanced lattice design in a feeding trial with swine in which the nine rations were compared. They found that the efficiency of the design relative to a randomized complete block design was 121 percent.

This is only a brief history of some of the early uses of lattice designs. For more information of review and uses of these designs see Experimental Design Theory and Application by W. T. Federer (1955).

Definition

The balanced lattice design is an incomplete block design that is characterized by the following basic features:

1. The number of treatments (t) must be an exact square.
2. The block size (k) is equal to the square root of the number of treatments. ($t = k^2$)
3. To achieve balance, the number of replications (r) is one more than the block size ($r = k+1$).
4. Each treatment occurs together in the same block with every other treatment exactly once. ($\lambda = 1$)

Field Arrangement and Randomization

Generally, blocks should consist of units that are as homogeneous as possible. In addition, blocks in the same replication should be as similar as possible to maximize the variation among replications. This will result in increased precision if the experiment is analyzed as a randomized block design.

Randomization proceeds as follows :

1. Randomize the order of blocks within a replication. A separate randomization is used in each replication.

2. Randomize the order of treatments within blocks.

The example of field arrangement and randomization of a 3 x 3 balanced lattice is as following:

The basic plan for 3 x 3 Balanced Lattice (t = 9, k = 3, r = 4, b = 12).

I	II	III	IV
(1) 1 2 3	(4) 1 4 7	(7) 1 5 9	(10) 1 8 6
(2) 4 5 6	(5) 2 5 8	(8) 7 2 6	(11) 4 2 9
(3) 7 8 9	(6) 3 6 9	(9) 4 8 3	(12) 7 5 3

where Block number is enclosed in parentheses.

Treatments are indicated by number within the blocks.

Step 1 :

Divide the experimental area into r = (k+1) replications. Each replication contains $t = k^2$ experimental plots. For example, the experimental area is divided into r = 4 replications, and each replication contains t = 9 experimental plots as shown in Figure 1.

Step 2 :

Divide each replication into k incomplete blocks. Each block contains k experimental plots. For example, each replication is divided into k = 3 incomplete blocks, each incomplete block contains k = 3 experimental plots as shown in Figure 1.

Figure 1 : Division of the experimental area, consisting of 36 plots (1, 2,..., 36) into four replications. Each replication contains three incomplete blocks of three plots in each block.

Block 1	1	2	3
2	4	5	6
3	7	8	9

Replication I

Block 1	10	11	12
2	13	14	15
3	16	17	18

Replication II

Block 1	19	20	21
2	22	23	24
3	25	26	27

Replication III

Block 1	28	29	30
2	31	32	33
3	34	35	36

Replication IV

Step 3 :

Randomize the replication arrangement of the basic plan: For example, suppose a table of random numbers method is applied:

- Select four three-digit random numbers: 532, 420, 861, 543.
- Rank the four three-digit random numbers:

Random numbers	Sequence	Rank
532	1	2
420	2	1
861	3	4
543	4	3

- Use the sequence to represent the existing replication number of the basic plan and the rank to represent the replication number of the new plan. Hence, the first replication of the basic plan (sequence = 1) becomes the second replication of the new plan (rank = 2), the second replication of the basic plan becomes the first replication of the new plan, and so on. Accordingly, the new plan is shown as Figure 2.

Figure 2 : Randomize the replication arrangement of the basic plan.

Incomplete Block Number	Treatment Number			
	Rep. I	Rep. II	Rep. III	Rep. IV
1	1 4 7	1 2 3	1 8 6	1 5 9
2	2 5 8	4 5 6	4 2 9	7 2 6
3	3 6 9	7 8 9	7 5 3	4 8 3

Step 4 :

Randomize the incomplete blocks with each replication. For example, the same randomization method used in step 3 is used to randomly reassign three incomplete blocks in each of the four replications. For instance, after four independent randomization processes, the reassigned incomplete blocks yielded :

Incomplete Block Number in the Basic plan	Reassigned Incomplete Block Number in New Plan			
	Rep. I	Rep. II	Rep. III	Rep. IV
1	3	2	3	1
2	2	1	1	3
3	1	3	2	2

From the above table, for replication I, block 1 of the basic plan becomes block 3 of the new plan, block 2 retains the same position, and block 3 of the basic plan becomes block1 of the new plan. For replication II, III, and IV are reassigned as well. Consequently, the new plan at this step is shown as Figure 3.

Figure 3 : Randomize the incomplete blocks with each replication.

Incomplete Block Number	Treatment Number			
	Rep. I	Rep. II	Rep. III	Rep. IV
1	3 6 9	4 5 6	7 5 3	1 5 9
2	2 5 8	1 2 3	1 8 6	4 8 3
3	1 4 7	7 8 9	4 2 9	7 2 6

Step 5 :

Randomize the treatment arrangement within each incomplete block. That is, randomly reassign the three treatments in each of the 12 incomplete blocks, using the same randomization method used in step 3 and step 4. For example, after 12 independent randomization processes, the reassigned treatment sequences yielded:

Treatment Sequence in Basic Plan	Reassigned Treatment Sequence in New Plan					
	Rep. I			Rep. II		
	Block 1	Block 2	Block 3	Block 1	Block 2	Block 3
1	2	3	2	2	3	3
2	3	2	3	1	2	2
3	1	1	1	3	1	1

Treatment Sequence in Basic Plan	Reassigned Treatment Sequence in New Plan					
	Rep. III			Rep. IV		
	Block 1	Block 2	Block 3	Block 1	Block 2	Block 3
1	3	3	1	1	3	2
2	2	1	2	3	1	3
3	1	2	3	2	2	1

At this step, for incomplete block 1 of replication I, treatment sequence 1 of the basic plan (treatment 3) becomes treatment sequence 2 of the new plan, treatment sequence 2 of the basic plan (treatment 6) becomes treatment sequence 3 of the new plan, and treatment sequence 3 of the basic plan (treatment 9) becomes treatment sequence 1 of the new plan, and so on. Thus, the new plan at this step is shown as Figure 4.

Figure 4 : Randomize the treatment arrangement within each incomplete block

Incomplete Block Number	Treatment Number			
	Rep. I	Rep. II	Rep. III	Rep. IV
1	9 3 6	5 4 6	3 5 7	1 5 9
2	8 5 2	3 2 1	8 6 1	8 3 4
3	7 1 4	9 8 7	4 2 9	6 7 2

Based on the basic plan of 3 x 3 balanced lattice, the field arrangement and randomization result in the new plan as shown in the following:

The new plan for 3 x 3 Balanced Lattice (t = 9, k = 3, r = 4, b = 12).

I	II	III	IV
(1) 9 3 6	(4) 5 4 6	(7) 3 5 7	(10) 1 9 5
(2) 8 5 2	(5) 3 2 1	(8) 8 6 1	(11) 8 3 4
(3) 7 1 4	(6) 9 8 7	(9) 4 2 9	(12) 6 7 2

where Block number is enclosed in parentheses.

Treatments are indicated by number within the blocks.

Analysis

Assume that the design is for k^2 treatments with $k + 1$ replicates. There are k blocks of k units each in each replication. The data table associated with a balance lattice design is shown in Table 1.

The linear model which describes an observation from a balanced lattice design, and on which the analysis is based, is,

The model :

$$Y_{ij(l)} = \mu + \rho_i + \tau_j + \beta_l + \varepsilon_{ij(l)}$$

Where

$Y_{ij(l)}$ is the response of the j^{th} treatment in the l^{th} block of the i^{th} replication.

μ is the overall mean.

ρ_i is the effect of the i^{th} replication.

τ_j is the effect of the j^{th} treatment.

β_l is the effect of the l^{th} block.

$\varepsilon_{ij(l)}$ is a random error where $\varepsilon_{ij(l)} \sim N(0, \sigma^2)$.

Table 1 : Data Table for a Balanced Lattice Design

Replication	Block	Response				Sum
1	1	$Y_{1,1,(1)}$	$Y_{1,2,(1)}$	...	$Y_{1,k,(1)}$	B_{11}
	2	$Y_{1,k+1,(2)}$	$Y_{1,k+2,(2)}$	...	$Y_{1,k+k,(2)}$	B_{12}
	...			...		...
	k	$Y_{1,k^2-k,(k)}$	...	...	$Y_{1,k^2,(k)}$	B_{1k}
Sum						R_1
2	1	$Y_{2,1,(1)}$	$Y_{2,2,(1)}$	...	$Y_{2,k,(1)}$	B_{21}
	2	$Y_{2,k+1,(2)}$	$Y_{2,k+2,(2)}$	...	$Y_{2,k+k,(2)}$	B_{22}
	...			...		...
	k	$Y_{2,k^2-k,(k)}$	...	...	$Y_{2,k^2,(k)}$	B_{2k}
Sum						R_2
	...			...		...
k+1	1	$Y_{k+1,1,(1)}$	$Y_{k+1,2,(1)}$	...	$Y_{k+1,k,(1)}$	$B_{k+1,1}$
	2	$Y_{k+1,k+1,(2)}$	$Y_{k+1,k+2,(2)}$	...	$Y_{k+1,k+k,(2)}$	$B_{k+1,2}$
	...			...		...
	k	$Y_{k+1,k^2-k,(k)}$	...	...	$Y_{k+1,k^2,(k)}$	$B_{k+1,k}$
Sum						R_{k+1}

Where B_{il} = Sum of the responses over k units in the l^{th} block of replication i

R_i = Sum of the responses in the i^{th} replication = $\sum_l B_{il}$

The Analysis of Variance

An analysis of variance table for a balanced lattice design and the sum of squares in the analysis of variance table are computed using the following summary of the notation and equations presented in Table 2.

Analysis of Variance of a Balanced Lattice Design

Source	df	ss	
Replication	k	SSR	
Treatment (unadjusted)	$k^2 - 1$	SSTr	
Block (adjusted)	$k^2 - 1$	SSB	E_b
Intrablock error	$(k - 1)(k^2 - 1)$	SSE	E_e
Total	$k^3 + k^2 + 1$	SST	

Table 2 : Treatment Totals and Adjustments for a Balanced Lattice Design

Treatment	Total (T_j)	Block (B_j)	Weight (W_j)
1	T_1	B_1	W_1
2	T_2	B_2	W_2
...	...	...	...
k^2	T_{k^2}	B_{k^2}	W_{k^2}
Sum	**G**	**kG**	**0**

Where

T_j = Sum of the responses for treatment j = $\sum_i y_{ij(l)}$

G = Grand total of all responses = $\sum_i R_i = \sum_j T_j$

B_j = Sum of the B_{il} for all blocks in which the j^{th} treatment occurs.

$$W_j = kT_j - (k-1)B_j + G$$

$$\sum_j W_j = 0$$

The sum of squares in analysis of variance table are computed using the following equations:

a correction term $C = G^2 / (k^3 + k^2)$

$$SST = \sum_i \sum_j Y_{ij(l)}{}^2 - C$$

$$SSR = \frac{1}{k^2} \sum_i R_i^2 - C$$

$$SSTr(Unadjusted) = \frac{1}{(k+1)} \sum_j T_j^2 - C$$

$$SSB(adjusted) = \frac{1}{k^3(k+1)} \sum_j W_j^2$$

$$SSE = SST - SSR - SSTr - SSB$$

$$E_b = \frac{SSB}{(k^2 - 1)} \quad , \quad E_e = \frac{SSE}{(k-1)(k^2-1)}$$

The next step is to compare E_b with E_e.

If $E_b \le E_e$ the adjustment for blocks will have no effect on the analysis. In this case the blocking restrictions are ignored and the data are analyzed as if they had come from a randomized block design with replications as blocks.

If $E_b > E_e$ the blocking is effective. In this case adjusted treatment totals and means are computed. First an adjustment factor (A) is computed:

$$A = (E_b - E_e) / k^2 E_b.$$

Then the adjusted treatment totals ($\hat{T}_j$) are computed : $\hat{T}_j = T_j + AW_j$

The effective error mean square ($E_e^{'}$) is calculated as :

$$E_e^{'} = E_e(1 + kA) \quad \text{with} \quad (k-1)(k^2-1) \ \text{d.f.}$$

To test the significance of the differences among the adjusted treatment means compute an adjusted treatment sum of squares (SSTr (adjusted)),

$$SSTr(adjusted) = \frac{1}{(k+1)} \sum_j \hat{T}_j^2 .$$

Then, the adjusted treatment mean square (E_t) is

$$E_t = SSTr(adjusted) / (k^2 - 1)$$

and an approximate F for testing significance is

$$F = E_t / E_e^{'} \quad \text{with} \quad [k^2 - 1, (k-1)(k^2-1)] \ \text{d.f.}$$

The adjusted treatment means ($\hat{\bar{y}}_j$) are $\hat{\bar{Y}}_j = \hat{T}_j / (k+1)$.
The estimated variance of an adjusted mean ($V(\hat{\bar{Y}})$) is $V(\hat{\bar{Y}}) = E_e^{'} / (k+1)$

and the estimated variance of the difference between adjusted means is

$$V(\hat{\bar{d}}) = 2E_e^{'} / (k+1).$$

To determine the relative precision of the balanced lattice, first the pooled error of a randomized block design is calculated (E_{rb}) as :

$$E_{rb} = (SSB + SSE) / k(k^2 - 1).$$

Then the precision of the balanced latticed lattice relative to that of a randomized block design is $\%\ relative \quad precision = (E_{rb} / E_e^{'})100.$

Example 1 : An example of a balanced lattice design with nine treatments.
(number of treatments (t) = 9, block size (k) = 3, number of replications (r) = 4, and number of blocks (b) = 12).

Data of R. E. Comstock, W. J. Peterson, and H. A. Stewart's paper (1948).
" An Application of the Balanced Lattice Design in a Feeding Trial with Swine" as shown in Table 3.

Table 3 : Data of R. E. Comstock, W. J. Peterson, and H. A. Stewart's paper.

Rep. I					Rep. II				
Block				Totals	Block				Totals
1	(1)	(2)	(3)		4	(1)	(4)	(7)	
	2.20	1.84	2.18	6.12		1.19	1.20	1.15	3.54
2	(4)	(5)	(6)		5	(2)	(5)	(8)	
	2.05	0.85	1.86	4.76		2.26	1.07	1.45	4.78
3	(7)	(8)	(9)		6	(3)	(6)	(9)	
	0.73	1.60	1.76	4.09		2.12	2.03	1.63	5.78
				15.07					14.10

Rep. III					Rep. IV				
Block				Totals	Block				Totals
7	(1)	(5)	(9)		10	(1)	(6)	(8)	
	1.81	1.16	1.11	4.08		1.77	1.57	1.43	4.77
8	(2)	(6)	(7)		11	(2)	(4)	(9)	
	1.76	2.16	1.80	5.72		1.50	1.60	1.42	4.52
9	(3)	(4)	(8)		12	(3)	(5)	(7)	
	1.71	1.57	1.13	4.41		2.04	0.93	1.78	4.75
				14.21					14.04

The values enclosed in parentheses correspond to the treatment numbers.

The results of statistical analysis of the data by using SAS follow. The SAS code and output are shown in Appendix I .

Table 4 : Analysis of Variance of R. E. Comstock, W. J. Peterson, and H. A. Stewart's data.

Source	df	SS	
Replication	3	0.0774	
Treatment (Unadj.)	8	3.2261	
Block (adj.)	8	1.4206	$E_b = 0.1776$
Intrablock error	16	1.2368	$E_e = 0.0773$
Total	35	5.9609	

The interpretation

From the ANOVA table, E_b is compared with E_e.

In this example, E_b (0.1776) > E_e (0.0773), so the blocking is effective. Thus, the adjusted treatment totals are computed as shown in Table 5.

Table 5 : Treatment Totals and Adjustment for a Balance Lattice Design

Treatment	Total (T_j)	Block (B_j)	Weight (W_j)	Adjusted Treatment ($\hat{T}_j$)
1	6.97	18.61	3.89	7.216
2	7.36	21.24	-5.46	7.016
3	8.05	21.16	-3.07	7.856
4	6.42	17.23	7.76	6.908
5	4.01	18.37	-4.03	3.756
6	7.62	21.03	-3.84	7.38
7	5.46	18.1	1.4	5.548
8	5.61	18.05	2.05	5.74
9	5.92	18.47	1.3	6
Sum	G = 57.42	172.26	0	57.42

Where $A = (E_b - E_e) / k^2 E_b$

$$= (0.1776 - 0.0773) / 3^2 (0.1776) = 0.06275.$$

To test the significance of differences among the adjusted treatment means:

Testing $\tau_1 = \tau_2 = \ldots = \tau_{k^2}$.

To test the null hypothesis $\tau_1 = \tau_2 = \ldots = \tau_9$, the statistic F is computed

as $F = \dfrac{E_t}{E'_e}$ *with* $[k^2 - 1, (k-1)(k^2 - 1)]$ *d.f.*

$$E_t = \frac{SST\,(adj.)}{(k^2 - 1)} = \frac{3.17309}{(3^2 - 1)} = 0.3966362.$$

$$E'_e = E_e(1 + kA) = 0.0773[1+3(0.06275)] = 0.09185.$$

Thus $F = \dfrac{0.3966362}{0.09185} = 4.3182 \approx 4.32$ with (8, 16) d.f.

and the associated p–value is 0.0062. Thus, there is sufficient evidence to reject the null hypothesis at an $\alpha = 0.05$ *level*. Also, the final analysis of variance is shown in Table 6.

Table 6 : Analysis of Variance Table

Source	df	SS	MS	F	P-value
Replication	3	0.0774			
Treatment (Unadj.)	8	3.2261			
Block (adj.)	8	1.4206	0.1776		
Intrablock error	16	1.2368	0.0773		
Treatment (adj.)	8	3.17309	0.39664	4.32	0.0062
Effective error	16		0.091852		
Total	35				

To estimate the precision relative to randomized block design,

$$\% \, relative \;\; precision \; = \left(\frac{E_{rb}}{E'_e} \right) 100 .$$

$$E_{rb} \;=\; (SSB + SSE) / k(k^2 - 1)$$

$$= \left(\frac{1.4206 + 1.2368}{3(3^2 - 1)} \right) = \frac{2.6574}{24} = 0.110725 .$$

Thus $\% \, relative \;\; precision \; = \left(\frac{0.110725}{0.091852} \right) 100 \;=\; 120.55 \;\approx 121\% .$

This means that the efficiency of the 3 x 3 balanced lattice design relative to a randomized complete block design is 121 percent.

Example 2 : An example of a balanced lattice design with sixteen treatments. (number of treatment (t) = 16, block size (k) = 4, number of replications (r) = 5, and number of blocks (b) = 20).

Data of K. A. Gomez and A. A. Gomez (1976). Statistical Procedures for Agricultural Research second edition. as shown in Table 7.

Table 7 : Tiller Number per Square Meter from 16 Fertilizer Treatments.

Rep. I					
Block					Totals
1	(1)	(2)	(3)	(4)	
	147	152	167	150	615
2	(5)	(6)	(7)	(8)	
	127	155	162	172	616
3	(9)	(10)	(11)	(12)	
	147	100	192	177	616
4	(13)	(14)	(15)	(16)	
	155	195	192	205	747
					2595

Rep. II					
Block					Totals
5	(1)	(5)	(9)	(13)	
	140	165	182	152	639
6	(10)	(2)	(14)	(6)	
	97	155	192	142	586
7	(7)	(15)	(3)	(11)	
	155	182	192	192	721
8	(16)	(8)	(12)	(4)	
	182	207	232	162	783
					2729

Rep. III					
Block					Totals
9	(1)	(6)	(11)	(16)	
	155	162	177	152	646
10	(5)	(2)	(15)	(12)	
	182	130	177	165	654
11	(9)	(14)	(3)	(8)	
	137	185	152	152	626
12	(13)	(10)	(7)	(4)	
	185	122	182	192	681
					2607

Rep. IV					
Block					Totals
13	(1)	(14)	(7)	(12)	
	220	202	175	205	802
14	(13)	(2)	(11)	(8)	
	205	152	180	187	724
15	(5)	(10)	(3)	(16)	
	165	150	200	160	675
16	(9)	(6)	(15)	(4)	
	155	177	185	172	689
					2890

Rep. V					
Block					Totals
17	(1)	(10)	(15)	(8)	
	147	112	177	147	583
18	(9)	(2)	(7)	(16)	
	180	205	190	167	742
19	(13)	(6)	(3)	(12)	
	172	212	197	192	773
20	(5)	(14)	(11)	(4)	
	177	220	205	225	827
					2925

Notation : The values enclosed in parentheses correspond to the treatment numbers.

The results of statistical analysis of the data by using SAS follow. The SAS code and output are shown in Appendix II.

Table 8 : Analysis of Variance of K. A. Gomez and A. A. Gomez's data.

Source	**df**	**SS**	
Replication	4	5946.05	
Treatment (Unadj.)	15	26,994.35	
Block (adj.)	15	11,381.84	$E_b = 758.79$
Intrablock error	45	14,533.31	$E_e = 322.96$
Total	79	58855.55	

The interpretation

First E_b is compared with E_e.

In this case, E_b (758.79) > E_e(322.96), so the blocking is effective. Hence, the adjusted treatment totals are computed as shown in Table 9.

To test the significance of differences among the adjusted treatment means:

Testing $\tau_1 = \tau_2 = \ldots = \tau_{k^2}$.

To test the null hypothesis $\tau_1 = \tau_2 = \ldots = \tau_{16}$, the statistic F is computed

as $F = \dfrac{E_t}{E'_e}$ *with* $[k^2 - 1, (k-1)(k^2-1)]\ d.f.$

$$E_t = \frac{SST\,(adj.)}{(k^2-1)} = \frac{24{,}001.55}{(4^2-1)} = 1600.1.$$

$$E'_e = E_e(1+kA) = 14{,}533.31[1+4(0.035898)] = 369.34.$$

Thus $F = \dfrac{1600.10}{369.34} = 4.33236 \approx 4.33$ with (15, 45) d.f.

and the associated p-value is 0.000065. Thus, there is sufficient evidence to reject the null hypothesis at an $\alpha = 0.05$ *level*. That is, at least one treatment (Fertilizer) affects the response differently from the other treatments. Consequently, the final analysis of variance of the data is shown in Table 10.

Table 9 : Treatment Totals and Adjustment for a Balanced Lattice Design.

Treatment	Total (T_j)	Block (B_j)	Weight (W_j)	Adjusted Treatment ($\hat{T}_j$)
1	809	3,286	552	829
2	794	3,322	312	805
3	908	3,411	323	920
4	901	3,596	-630	878
5	816	3,411	-45	814
6	848	3,310	588	869
7	864	3,562	-608	842
8	865	3,332	546	885
9	801	3,312	390	815
10	581	3,141	365	594
11	946	3,534	-140	941
12	971	3,628	-510	953
13	869	3,564	-598	848
14	994	3,588	-218	986
15	913	3,394	428	928
16	866	3,593	-755	839
Sum	G = 13,746	54,984	0	13,746.01

Where $A = (E_b - E_e) / k^2 E_b$

$$= (758.79 - 322.96) / 4^2 (758.79) = 0.035898.$$

Table 10 : Analysis of Variance Table

Source	df	SS	MS	F	P-value
Replication	4	5,946.05			
Treatment (Unadj.)	15	26,994.35			
Block (adj.)	15	11,381.84	758.79		
Intrablock error	45	14,533.31	322.96		
Treatment (adj.)	15	24,001.55	1600.1	4.33	0.000065
Effective error	45		369.34		
Total	79	58,855.55			

To estimate the precision relative to randomized complete block design,

$$\%\ relative\ \ precision\ =\ \left(\frac{E_{rb}}{E_e'}\right)100.$$

$$E_{rb}\ =\ (SSB+SSE)/k(k^2-1)$$

$$=\left(\frac{11{,}381.84+14{,}533.31}{4(4^2-1)}\right)=\frac{25{,}915.15}{60}\ =\ 431.919.$$

Thus $\%\ relative\ \ precision\ =\ \left(\frac{431.919}{369.34}\right)100\ =\ 116.94\ \approx\ 117\%.$

This means that the 4 x 4 balanced lattice design have increased the experimental precision by 17 percent relative to a randomized complete block design.

Comments

The lattice designs are useful for analyzing a large number of treatments within relatively small blocks, and each pair of treatments is compared with the same degree of precision because each treatment appears together in the same block with every other treatment an equal number of times. However, to obtain a balanced lattice, some restrictions on the number of treatments and the number of blocks in the design are required. As a result, balanced lattice designs are not available for 36, 100, 144 treatments.

In this writing project, the purpose is to analyze data from a balanced lattice design using SAS. The two examples of 9, and 16 treatments were used to illustrate and develop a better understanding. Also the methods and analysis in this paper showed the major methods of analysis of lattice designs including SAS code and output. Nevertheless, there are several types of lattice designs excluded in the paper such as the simple lattice, lattice square, triple lattice, and rectangular lattice which may be more complicated than the examples shown in this paper. Furthermore, in practice the experimental method could be more complex than the designs given in these examples. Nonetheless , the methods discussed in the paper could be a foundation of analysis of these more complex lattice designs.

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APPENDIX I

SAS CODE FOR EXAMPLE 1:

```
*********        LATTICE  DESIGN        *********
******  for  NUMBER  OF  TREATMENT  = 9  ******

*****  PART I : FINDING Eb, Ee AND Adjusted Treatment Means  *****

dm 'log;clear;out;clear;';
options ps = 60 ls = 64 nodate nonumber;
data in;
     infile 'pig';
     input group block treatmnt response;
proc print data = in;
proc lattice data = in;
     var response;
run;

*****  PART II : ADJUSTED  SUM  OF  SQUARE  TREATMENT  WHEN  Eb > Ee  *****

data adjtrt;
     array y(9) y1-y9;
     array t(9) t1-t9;
     array tt(9) tt1-tt9;
     sum = 0; s2 = 0; k = 3;

         do i = 1 to 9;
         input y(i) @@;
         t(i) =(k+1)*y(i);
         tt(i) = t(i)*t(i);
         sum = sum + t(i);
         s2  = s2 + tt(i);
         end;
drop i;
     C = (sum*sum)/(k*k*(k+1));
     SSTradj = (s2/(k+1))- C;
     Eb = .1776; Ee = .0773;
     A = (Eb - Ee)/(k*k*Eb);
     Eeff = Ee*(1+k*A);
     F = SSTradj/(Eeff*(k*k-1));
     pvalue = 1 - probf(F,k*k-1,((k-1)*(k*k-1)));
output;

*****  INPUT THE ADJUSTED TREATMENT MEANS GOT FROM PART I  *****

cards;
1.804  1.754  1.964  1.727  0.939  1.845  1.387  1.435  1.500
;
proc print data = adjtrt;

run;
```

SAS OUTPUT FOR EXAMPLE 1:

****** DATA *****

OBS	GROUP	BLOCK	TREATMNT	RESPONSE
1	1	1	1	2.20
2	1	1	2	1.84
3	1	1	3	2.18
4	1	2	4	2.05
5	1	2	5	0.85
6	1	2	6	1.86
7	1	3	7	0.73
8	1	3	8	1.60
9	1	3	9	1.76
10	2	1	1	1.19
11	2	1	4	1.20
12	2	1	7	1.15
13	2	2	2	2.26
14	2	2	5	1.07
15	2	2	8	1.45
16	2	3	3	2.12
17	2	3	6	2.03
18	2	3	9	1.63
19	3	1	1	1.81
20	3	1	5	1.16
21	3	1	9	1.11
22	3	2	2	1.76
23	3	2	6	2.16
24	3	2	7	1.80
25	3	3	3	1.71
26	3	3	4	1.57
27	3	3	8	1.13
28	4	1	1	1.77
29	4	1	6	1.57
30	4	1	8	1.43
31	4	2	2	1.50
32	4	2	4	1.60
33	4	2	9	1.42
34	4	3	3	2.04
35	4	3	5	0.93
36	4	3	7	1.78

Analysis of Variance for Variable RESPONSE

Source	DF	SS	MS
Replications	3	0.0774	0.0258
Blocks within Replications (Adj.)	8	1.4206	0.1776
Component B	8	1.4206	0.1776
Treatments (Unadj.)	8	3.2261	0.4033
Intra Block Error	16	1.2368	0.0773
Randomized Complete Block Error	24	2.6574	0.1107
Total	35	5.9609	0.1703

Variance of Means in Same Block 0.045925

LSD at .01 Level 0.625929

LSD at .05 Level 0.454300

The Efficiency of the Experiment Relative to Randomized Complete Block is 120.55.

Adjusted Treatment Means

1	1.804	4	1.727	6	1.845	8	1.435
2	1.754	5	0.939	7	1.387	9	1.500
3	1.964						

OBS	Y1	Y2	Y3	Y4	Y5	Y6	Y7	Y8	Y9
1	1.804	1.754	1.964	1.727	0.939	1.845	1.387	1.435	1.5

OBS	T1	T2	T3	T4	T5	T6	T7	T8	T9	TT1
1	7.216	7.016	7.856	6.908	3.756	7.38	5.548	5.74	6	52.0707

OBS	TT2	TT3	TT4	TT5	TT6	TT7	TT8
1	49.2243	61.7167	47.7205	14.1075	54.4644	30.7803	32.9476

OBS	TT9	SUM	S2	K	C	SSTRADJ	EB	EE	A
1	36	57.42	379.032	3	91.5849	3.17309	0.1776	0.0773	0.062750

OBS	EEFF	F	PVALUE
1	0.091852	4.31822	.0062090

APPENDIX II

SAS CODE FOR EXAMPLE 2:

```
*********        LATTICE  DESIGN       *********
******  for  NUMBER  OF  TREATMENT = 16   ******

*****  PART I : FINDING Eb, Ee AND Adjusted Treatment Means  *****

dm 'log;clear;out;clear;';
options ps = 60 ls = 64 nodate nonumber;
data in;
     infile 'tiller';
     input group block treatmnt response;
proc print data = in;
proc lattice data = in;
     var response;
run;

*****  PART II : ADJUSTED  SUM  OF SQUARE TREATMENT WHEN  Eb > Ee *****

data adjtrt;
     array y(16) y1-y16;
     array t(16) t1-t16;
     array tt(16) tt1-tt16;
     sum = 0; ss2 = 0; k = 4;
        do i = 1 to 16;
        input y(i) @@;
        t(i) =(k+1)*y(i);
        tt(i) = t(i)*t(i);
        sum = sum + t(i);
        ss2 = ss2 + tt(i);
        end;
drop i;
     C = (sum*sum)/(k*k*(k+1));
     SSTradj = (ss2/(k+1))- C;
     Eb = 758.7892; Ee = 322.9625;
     A = (Eb - Ee)/(k*k*Eb);
     Eeff = Ee*(1+k*A);
     F = SSTradj/(Eeff*(k*k-1));
     pvalue = 1 - probf(F,k*k-1,((k-1)*(k*k-1)));
output;

*****  INPUT THE ADJUSTED TREATMENT MEANS GOT FROM PART I  *****
cards;
165.763  161.040  183.919  175.677
162.877  173.822  168.435  176.920
163.000  118.821  188.195  190.538
169.507  197.235  185.673  167.779
;
proc print data = adjtrt;
run;
```

SAS OUTPUT FOR EXAMPLE 2:

***** DATA *****

OBS	GROUP	BLOCK	TREATMNT	RESPONSE
1	1	1	1	147
2	1	1	2	152
3	1	1	3	167
4	1	1	4	150
5	1	2	5	127
6	1	2	6	155
7	1	2	7	162
8	1	2	8	172
9	1	3	9	147
10	1	3	10	100
11	1	3	11	192
12	1	3	12	177
13	1	4	13	155
14	1	4	14	195
15	1	4	15	192
16	1	4	16	205
17	2	1	1	140
18	2	1	5	165
19	2	1	9	182
20	2	1	13	152
21	2	2	10	97
22	2	2	2	155
23	2	2	14	192
24	2	2	6	142
25	2	3	7	155
26	2	3	15	182
27	2	3	3	192
28	2	3	11	192
29	2	4	16	182
30	2	4	8	207
31	2	4	12	232
32	2	4	4	162
33	3	1	1	155
34	3	1	6	162
35	3	1	11	177
36	3	1	16	152
37	3	2	5	182
38	3	2	2	130
39	3	2	15	177
40	3	2	12	165
41	3	3	9	137
42	3	3	14	185
43	3	3	3	152
44	3	3	8	152
45	3	4	13	185
46	3	4	10	122

***** DATA (continue) *****

OBS	GROUP	BLOCK	TREATMNT	RESPONSE
47	3	4	7	182
48	3	4	4	192
49	4	1	1	220
50	4	1	14	202
51	4	1	7	175
52	4	1	12	205
53	4	2	13	205
54	4	2	2	152
55	4	2	11	180
56	4	2	8	187
57	4	3	5	165
58	4	3	10	150
59	4	3	3	200
60	4	3	16	160
61	4	4	9	155
62	4	4	6	177
63	4	4	15	185
64	4	4	4	172
65	5	1	1	147
66	5	1	10	112
67	5	1	15	177
68	5	1	8	147
69	5	2	9	180
70	5	2	2	205
71	5	2	7	190
72	5	2	16	167
73	5	3	13	172
74	5	3	6	212
75	5	3	3	197
76	5	3	12	192
77	5	4	5	177
78	5	4	14	220
79	5	4	11	205
80	5	4	4	225

Analysis of Variance for Variable RESPONSE

Source	DF	SS	MS
Replications	4	5946.0500	1486.5125
Blocks within Replications (Adj.)	15	11381.8375	758.7892
Component B	15	11381.8375	758.7892
Treatments (Unadj.)	15	26994.3500	1799.6233
Intra Block Error	45	14533.3125	322.9625
Randomized Complete Block Error	60	25915.1500	431.9192
Total	79	58855.5500	745.0070

Variance of Means in Same Block 147.735037

LSD at .01 Level 32.690911

LSD at .05 Level 24.480682

The Efficiency of the Experiment Relative to Randomized Complete Block is 116.94.

Adjusted Treatment Means

1	165.763	5	162.877	9	163.000	13	169.507
2	161.040	6	173.822	10	118.821	14	197.235
3	183.919	7	168.435	11	188.195	15	185.673
4	175.677	8	176.920	12	190.538	16	167.779

OBS	Y1	Y2	Y3	Y4	Y5	Y6	Y7
1	165.763	161.04	183.919	175.677	162.877	173.822	168.435

OBS	Y8	Y9	Y10	Y11	Y12	Y13	Y14	Y15
1	176.92	163	118.821	188.195	190.538	169.507	197.235	185.673

OBS	Y16	T1	T2	T3	T4	T5	T6	T7
1	167.779	828.815	805.2	919.595	878.385	814.385	869.11	842.175

OBS	T8	T9	T10	T11	T12	T13	T14	T15
1	884.6	815	594.105	940.975	952.69	847.535	986.175	928.365

OBS	T16	TT1	TT2	TT3	TT4	TT5
1	838.895	686934.30	648347.04	845654.96	771560.21	663222.93

```
OBS     TT6        TT7        TT8        TT9       TT10       TT11

 1  884.6 815 594.105 940.975 952.69 847.535 986.175 928.365

OBS    T16        TT1        TT2        TT3        TT4        TT5

 1  838.895 686934.30 648347.04 845654.96 771560.21 663222.93

OBS     TT6        TT7        TT8        TT9       TT10       TT11

 1  755352.19 709258.73 782517.16 664225 352960.75 885433.95

OBS     TT12       TT13       TT14       TT15       TT16        SUM

 1  907618.24 718315.58 972541.13 861861.57 703744.82 13746.01

OBS      SS2      K       C        SSTRADJ      EB       EE        A

 1  11929548.57 4 2361908.17 24001.55 758.789 322.963 0.035898

OBS    EEFF         F          PVALUE

 1  369.338     4.33236     .000065350
```

Appendix III : Plans for Lattice designs

Notation:
1. t = number of treatments.
2. k = block size.
3. r = number of replications.
4. b = number of blocks.

Block number is enclosed in parentheses.

Replication number is in Roman numerals.

Treatments are indicated by number within the blocks.

1. Plan for 3 x 3 Balanced Lattice

t = 9, k = 3, r = 4, b = 12

I				II			
(1)	1	2	3	(4)	1	4	7
(2)	4	5	6	(5)	2	5	8
(3)	7	8	9	(6)	3	6	9

III				IV			
(7)	1	5	9	(10)	1	8	6
(8)	7	2	6	(11)	4	2	9
(9)	4	8	3	(12)	7	5	3

2. Plan for 4 x 4 Balanced Lattice

t = 16, k = 4, r = 5, b = 20

I					II					III				
(1)	1	2	3	4	(5)	1	5	9	13	(9)	1	6	11	16
(2)	5	6	7	8	(6)	2	6	10	14	(10)	5	2	15	12
(3)	9	10	11	12	(7)	3	7	11	15	(11)	9	14	3	8
(4)	13	14	15	16	(8)	4	8	12	16	(12)	13	10	7	4

IV				
(13)	1	14	7	12
(14)	13	2	11	8
(15)	5	10	3	16
(16)	9	6	15	4

V				
(17)	1	10	15	8
(18)	9	2	7	16
(19)	13	6	3	12
(20)	5	14	11	4

3. Plan for 5 x 5 Balanced Lattice

t = 25, k = 5, r = 6, b = 30

I					
(1)	1	2	3	4	5
(2)	6	7	8	9	10
(3)	11	12	13	14	15
(4)	16	17	18	19	20
(5)	21	22	23	24	25

II					
(6)	1	6	11	16	21
(7)	2	7	12	17	22
(8)	3	8	13	18	23
(9)	4	9	14	19	24
(10)	5	10	15	20	25

III					
(11)	1	7	13	19	25
(12)	21	2	8	14	20
(13)	16	22	3	9	15
(14)	11	17	23	4	10
(15)	6	12	18	24	5

IV					
(16)	1	12	23	9	20
(17)	16	2	13	24	10
(18)	6	17	3	14	25
(19)	21	7	18	4	15
(20)	11	22	8	19	5

V					
(21)	1	17	8	24	15
(22)	11	2	18	9	25
(23)	21	12	3	19	10
(24)	6	22	13	4	20
(25)	16	7	23	14	5

VI					
(26)	1	22	18	14	10
(27)	6	2	23	19	15
(28)	11	7	3	24	20
(29)	16	12	8	4	25
(30)	21	17	13	9	5