EQUATIONS FOR PROBABILITY & MATHEMATICAL STATISTICS

Robert J. Boik, November 12, 2008

1 Definitions & Theorems

$\mathbb{N} \stackrel{\text{def}}{=} \{1, 2, \ldots, \infty\}$	$\mathbb{N}_0 \stackrel{\text{def}}{=} \{0, 1, 2, \ldots, \infty\}$
If $n \in \mathbb{N}_0$, then $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$	If $n \in \mathbb{N}_0$, then $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
If $r > 0$, then $\Gamma(r) \stackrel{\text{def}}{=} \int_0^\infty x^{r-1} e^{-x}\, dx$ and $r! \stackrel{\text{def}}{=} \Gamma(r+1)$	Integration by Parts $\int_a^b u\, dv = uv\Big\vert_a^b - \int_a^b v\, du$
Stirling's Approximation $r! \approx \sqrt{2\pi} r^{r+1/2} e^{-r}$	Permutations $(N)_m \stackrel{\text{def}}{=} N!/(N-m)!$
Combinations Binomial Coefficient $\binom{N}{m} = \binom{N}{m, N-m} \stackrel{\text{def}}{=} \frac{N!}{m!(N-m)!}$	Binomial Theorem If $N \in \mathbb{N}_0$, then $\sum_{i=0}^{N} \binom{N}{i} a^i b^{N-i} = (a+b)^N$
Multinomial Coefficient $\binom{N}{m_1, m_2, \ldots, m_k} \stackrel{\text{def}}{=} N! / \prod_{i=1}^{k} m_i!$ where $m_i \in \mathbb{N}_0$ and $\sum_{i=1}^{k} m_i = N$	Multinomial Theorem $\sum_{R} \binom{N}{m_1, m_2, \ldots, m_k} \prod_{i=1}^{k} a_i^{m_i} = \left(\sum_{i=1}^{k} a_i\right)^N$ where $N \in \mathbb{N}_0$, $R = \left\{(m_1, m_2, \ldots, m_k); m_i \in \mathbb{N}_0 \text{ and } \sum_{i=1}^{k} m_i = N\right\}$
DeMorgan's Law $\left(\bigcup_{i=1}^{k} A_i\right)^c = \bigcap_{i=1}^{k} A_i^c$	DeMorgan's Law $\left(\bigcap_{i=1}^{k} A_i\right)^c = \bigcup_{i=1}^{k} A_i^c$
$P(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$	$f_{X\vert Y}(x\vert y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$
$f_{X,Y}(x,y) = f_Y(y) f_{X\vert Y}(x\vert y)$	Bayes' Theorem $\Pr(E_k\vert F) = \frac{\Pr(F\vert E_k)\Pr(E_k)}{\sum_{i=1}^{n} \Pr(F\vert E_i)\Pr(E_i)}$, where $E_1, \ldots, E_n$ is a partition of Ω
$F^\circ = 32 + \frac{9}{5} C^\circ$	$C^\circ = \frac{5}{9}(F^\circ - 32)$

2 Series

Taylor's Theorem $g(t) = g(t_0) + \sum_{i=1}^{k-1} \left(\frac{\partial^i g(t)}{(\partial\, t)^i}\Big\vert_{t=t_0} \right) \frac{(t-t_0)^i}{i!} + R_k,$ where $R_k = \left(\frac{\partial^k g(t)}{(\partial\, t)^i}\Big\vert_{t=t^*} \right) \frac{(t-t_0)^k}{k!}$ and $t^* \in \begin{cases} (t, t_0) & \text{if } t_0 > t \\ (t_0, t) & \text{if } t_0 < t; \end{cases}$	
If $n \in \mathbb{N}_0$, then $\sum_{i=0}^{n} r^i = \begin{cases} n+1 & r = 1, \\ \frac{1-r^{n+1}}{1-r} & \text{otherwise;} \end{cases}$	$\sum_{i=0}^{\infty} r^i = \begin{cases} \infty & r \geq 1, \\ \text{undefined} & r \leq -1, \\ (1-r)^{-1} & \lvert r \rvert < 1; \end{cases}$
If $\lvert r \rvert < 1$ and $a > 0$, then $(1-r)^{-a} = \sum_{i=0}^{\infty} \binom{a-1+i}{i} r^i$	$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$
If $-1 < \epsilon \leq 1$, then $\ln(1+\epsilon) = \sum_{i=1}^{\infty} (-1)^{i+1} \frac{\epsilon^i}{i}$	If $-1 \leq \epsilon < 1$, then $\ln(1-\epsilon) = -\sum_{i=1}^{\infty} \frac{\epsilon^i}{i}$
$\sin(x) = \sum_{i=1}^{\infty} (-1)^{i+1} \frac{x^{2i-1}}{(2i-1)!}$	$\cos(x) = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i}}{(2i)!}$
Hyperbolic Sine $\sinh(x) \stackrel{\text{def}}{=} \frac{e^x - e^{-x}}{2} = \sum_{i=0}^{\infty} \frac{x^{2i+1}}{(2i+1)!}$	

3 Selected Derivatives

Note: a is a constant

$\frac{d\,a^x}{d\,x} = a^x \ln(a)$	$\frac{d\,\log_a x}{d\,x} = \frac{1}{x\ln(a)}$	$\frac{d\,\sin(x)}{d\,x} = \cos(x)$
$\frac{d\,\cos(x)}{d\,x} = -\sin(x)$	$\frac{d\,\tan(x)}{d\,x} = [\sec(x)]^2$	$\frac{d\,\cot(x)}{d\,x} = -[\csc(x)]^2$
$\frac{d\,\sec(x)}{d\,x} = \sec(x)\,\tan(x)$	$\frac{d\,\arcsin(x)}{d\,x} = \frac{1}{\sqrt{1-x^2}}$	$\frac{d\,\arccos(x)}{d\,x} = -\frac{1}{\sqrt{1-x^2}}$
$\frac{d\,\arctan(x)}{d\,x} = \frac{1}{1+x^2}$	$\frac{d\,\csc(x)}{d\,x} = -\csc(x)\,\cot(x)$	$\frac{d\,\csc^{-1}(x)}{d\,x} = -\frac{1}{x\sqrt{x^2-1}}$
$\frac{d\,\sec^{-1}(x)}{d\,x} = \frac{1}{x\sqrt{x^2-1}}$	$\frac{d\,\cot^{-1}(x)}{d\,x} = -\frac{1}{1+x^2}$	

4 Miscellaneous

Definition: $X_n \xrightarrow{\text{prob}} X$ means that $\lim_{n\to\infty} P(\lvert X_n - X\rvert < \varepsilon) = 1$ for any $\varepsilon > 0$
Definition: $X_n \xrightarrow{\text{qm}} X$ means that $\lim_{n\to\infty} \mathrm{E}\left[(X_n - X)^2\right] = 0$
Definition: $X_n \xrightarrow{\text{dist}} X$ means that $\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$ at all points of continuity of $F_X(x)$
Hölder: $\lvert \mathrm{E}(XY)\rvert \le \mathrm{E}(\lvert XY\rvert) \le (\mathrm{E}\lvert X\rvert^p)^{\frac{1}{p}}\,(\mathrm{E}\lvert Y\rvert^q)^{\frac{1}{q}};\ p>1,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1$
$\mathrm{Var}(S^2) = \left(\frac{\sigma^4}{n-1}\right)\left[\rho_4\left(\frac{n-1}{n}\right)+2\right]$, where ρ_4 is the standardized kurtosis coefficient.
$X_i \overset{\text{iid}}{\sim} f_X(x)$ continuous $\Longrightarrow f_{X_{(i)}}(x) = \binom{n}{i-1,1,n-i} F_X(x)^{i-1} f_X(x)\,[1-F_X(x)]^{n-i}$
$X_i \overset{\text{iid}}{\sim} f_X(x)$ continuous and $j>i \Longrightarrow f_{X_{(i)},X_{(j)}}(x_i,x_j) =$ $\binom{n}{i-1,1,j-i-1,1,n-j} F_X(x)^{i-1} f_X(x_i)\,[F_X(x_j)-F_X(x_i)]^{j-i-1}\, f_X(x_j)\,[1-F_X(x_j)]^{n-j}$
if $T \sim \mathrm{Poisson}(\theta)$, then $P_\theta(T \ge t) = P(U \le \theta)$, where $U \sim \mathrm{Gam}(t,1)$
if $T \sim \mathrm{Poisson}(\theta)$, then $P_\theta(T \le t) = 1 - P(U \le \theta)$, where $U \sim \mathrm{Gam}(t+1,1)$
if $T \sim \mathrm{Bin}(n,\theta)$, then $P_\theta(T \le t) = P(U \le 1-\theta)$, where $U \sim \mathrm{Beta}(n-t,t+1)$
Leibniz: $\frac{d}{d\theta}\int_{a(\theta)}^{b(\theta)} g(x,\theta)dx = g\left[b(\theta),\theta\right]\frac{d}{d\theta}b(\theta) - g\left[a(\theta),\theta\right]\frac{d}{d\theta}a(\theta) + \int_{a(\theta)}^{b(\theta)} \frac{d}{d\theta}g(x,\theta)dx$

5 Discrete Distributions

Distribution	Notation	pmf: $f(x)$	cdf: $F(x)$	$\mathrm{E}(X)$	$\mathrm{Var}(X)$	MGF: $M_X(t)$
Bernoulli	$X \sim \mathrm{Bern}(p)$ $p \in (0,1)$	$p^x(1-p)^{1-x}$ $x = 0, 1$	$\sum_{i=0}^{[x]} f(i),\ x \le 1$ $1,\ x > 1$	p	$p(1-p)$	$pe^t + 1 - p$
Binomial	$X \sim \mathrm{Bin}(n,p)$ $p \in (0,1)$	$\binom{n}{x} p^x (1-p)^{n-x}$ $x = 0, 1, \ldots, n$	$\sum_{i=0}^{[x]} f(i),\ x \le n$ $1,\ x > n$	np	$np(1-p)$	$(pe^t + 1 - p)^n$
Discrete Uniform	$X \sim \mathrm{DUnif}(N)$ $N \in \mathbb{N}$	$\frac{1}{N}$ $x = 1, 2, \ldots, N$	$0,\ x < 1$ $\frac{[x]}{N},\ 1 \le x \le N$ $1,\ x > N$	$\frac{N+1}{2}$	$\frac{N^2-1}{12}$	$\frac{e^t(1-e^{Nt})}{N(1-e^t)}$
Geometric # Trials	$X \sim \mathrm{Geom}(p)$ $p \in (0,1)$	$p(1-p)^{x-1}$ $x \in \mathbb{N}$	$0,\ x < 1$ $1 - (1-p)^{[x]},\ x \ge 1$	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$\frac{pe^t}{1-(1-p)e^t}$, $t < -\ln(1-p)$
Geometric # Failures	$X \sim \mathrm{Geom}^*(p)$ $p \in (0,1)$	$p(1-p)^x$ $x \in \mathbb{N}_0$	$0,\ x < 0$ $1 - (1-p)^{[x+1]},\ x \ge 0$	$\frac{1-p}{p}$	$\frac{1-p}{p^2}$	$\frac{p}{1-(1-p)e^t}$, $t < -\ln(1-p)$
Hyper-geometric	$X \sim \mathrm{HypG}(N,M,n)$ $N \in \mathbb{N}$ $M = 0, 1, \ldots, N$ $n = 1, 2, \ldots, N$	$\frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$ $x = 0, 1, \ldots, b$ $b = \min(n, M)$ $n + M - N \le x \le b$	$0,\ x < 0$ $\sum_{i=1}^{[x]} f(i),\ 0 \le x \le b$ $1,\ x > b$	np $p = \frac{M}{N}$	$np(1-p)$ $\times \frac{(N-n)}{(N-1)}$	NU
Log Series	$X \sim \mathrm{LogSer}(p)$ $p \in (0,1)$	$-\frac{(1-p)^x}{\ln(p)\, x}$ $x \in \mathbb{N}$	$0,\ x < 1$ $\sum_{i=1}^{[x]} f(i),\ x \ge 1$	$-\frac{(1-p)}{p\ln(p)}$	$-\frac{(1-p)}{p^2(\ln p)^2} \times$ $[1 - p + \ln p]$	$\frac{\ln\left[1-(1-p)e^t\right]}{\ln p}$, $t < (1-p)^{-1}$
Negative Binomial # Trials	$X \sim \mathrm{NBin}(r,p)$ $r > 0,\ p \in (0,1)$	$\binom{x-1}{r-1} p^r (1-p)^{x-r}$ $x = r, r+1, \ldots$	$0,\ x < r$ $\sum_{i=r}^{[x]} f(i),\ x \ge r$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$	$\left(\frac{pe^t}{1-(1-p)e^t}\right)^r$, $t < -\ln(1-p)$
Negative Binomial # Failures	$X \sim \mathrm{NBin}^*(r,p)$ $r > 0,\ p \in (0,1)$	$\binom{x+r-1}{x} p^r (1-p)^x$ $x \in \mathbb{N}_0$	$0,\ x < 0$ $\sum_{i=0}^{[x]} f(i),\ x \ge 0$	$\frac{r(1-p)}{p}$	$\frac{r(1-p)}{p^2}$	$\left(\frac{p}{1-(1-p)e^t}\right)^r$, $t < -\ln(1-p)$
Poisson	$X \sim \mathrm{Poisson}(\lambda)$ $\lambda > 0$	$\frac{e^{-\lambda}\lambda^x}{x!}$ $x \in \mathbb{N}_0$	$0,\ x < 0$ $\sum_{i=0}^{[x]} f(i),\ x \ge 0$	λ	λ	$e^{\lambda(e^t - 1)}$

NE: Does not exist
NU: Not useful

6 Continuous Distributions

Distribution	Notation	pdf: $f(x)$	cdf: $F(x)$	E(X)	Var(X)	MGF: $M_X(t)$
Beta	$X \sim \text{Beta}(\alpha, \beta)$ $\alpha > 0,\ \beta > 0$	$\frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}$ $0 \le x \le 1$ $B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$	$0,\ x < 0$ $\int_0^x f(x)\,dx,$ $0 \le x \le 1$ $1,\ x > 1$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta+1)} \times \frac{1}{(\alpha+\beta)^2}$	NU
Cauchy	$X \sim$ Cauchy(μ, σ) $\sigma > 0$	$\frac{1}{\pi\sigma\left[1+\left(\frac{x-\mu}{\sigma}\right)^2\right]}$	$\frac{1}{2} + \frac{\arctan\left(\frac{x-\mu}{\sigma}\right)}{\pi}$	DNE	DNE	DNE
Chi-Squared	$X \sim \chi^2_\nu$ $\nu > 0$	$\frac{x^{\frac{\nu}{2}-1}e^{-\frac{x}{2}}}{\Gamma\left(\frac{\nu}{2}\right)2^{\frac{\nu}{2}}}$ $x \ge 0$	$0,\ x < 0$ $\int_0^x f(u)\,du,\ x > 0$	ν	2ν	$(1-2t)^{-\frac{\nu}{2}},$ $t < \frac{1}{2}$
Exponential	$X \sim \text{Expon}(\beta)$ $\beta > 0$	$\frac{e^{-x/\beta}}{\beta}$ $x \ge 0$	$0,\ x < 0$ $1 - e^{-x/\beta},\ x \ge 0$	β	β^2	$(1-\beta t)^{-1},$ $t < \frac{1}{\beta}$
Exponential	$X \sim \text{Expon}^*(\beta)$ $\beta > 0$	$\beta e^{-\beta x}$ $x \ge 0$	$0,\ x < 0$ $1 - e^{-\beta x},\ x \ge 0$	$\frac{1}{\beta}$	$\frac{1}{\beta^2}$	$\frac{\beta}{\beta - t},$ $t < \beta$
Exponential: Two-Parameter	$X \sim \text{Expon}(\mu, \sigma)$ $\sigma > 0$	$\frac{e^{-\frac{(x-\mu)}{\sigma}}}{\sigma}$ $x \ge \mu$	$0,\ x < \mu$ $1 - e^{-\frac{(x-\mu)}{\sigma}},\ x \ge \mu$	$\mu + \sigma$	σ^2	$\frac{e^{\mu t}}{1-\sigma t},$ $t < \frac{1}{\sigma}$
F	$X \sim F_{\nu_1,\nu_2}$ $\nu_1 > 0,\ \nu_2 > 0$	$\left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}} \frac{x^{\frac{\nu_1}{2}-1}}{B\left(\frac{\nu_1}{2},\frac{\nu_2}{2}\right)} \times \left(1 + \frac{\nu_1 x}{\nu_2}\right)^{-\frac{1}{2}(\nu_1+\nu_2)},$ $x > 0$	$0,\ x < 0$ $\int_0^x f(u)\,du,\ x > 0$	$\frac{\nu_2}{\nu_2 - 2},$ $\nu_2 > 2$	$2\left(\frac{\nu_2}{\nu_2-2}\right)^2 \times \left[\frac{\nu_1+\nu_2-2}{\nu_1(\nu_2-4)}\right],$ $\nu_2 > 4$	DNE
Gamma	$X \sim \text{Gam}(\alpha, \beta)$ $\alpha > 0,\ \beta > 0$	$\frac{x^{\alpha-1}e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha}$ $x \ge 0$	$0,\ x < 0$ $\int_0^x f(u)\,du,\ x > 0$	$\alpha\beta$	$\alpha\beta^2$	$(1-\beta t)^{-\alpha},$ $t < \frac{1}{\beta}$
Gamma	$X \sim \text{Gam}^*(\alpha, \beta)$ $\alpha > 0,\ \beta > 0$	$\frac{x^{\alpha-1}\beta^\alpha e^{-\beta x}}{\Gamma(\alpha)}$ $x \ge 0$	$0,\ x < 0$ $\int_0^x f(u)\,du,\ x > 0$	$\frac{\alpha}{\beta}$	$\frac{\alpha}{\beta^2}$	$\left(\frac{\beta}{\beta - t}\right)^\alpha,$ $t < \beta$
Laplace: Double Exponential	$X \sim$ DExpon(μ, σ) $\sigma > 0$	$\frac{e^{-\frac{\lvert x-\mu\rvert}{\sigma}}}{2\sigma}$	$\frac{e^{\frac{x-\mu}{\sigma}}}{2},\ x \le \mu$ $1 - \frac{e^{-\frac{(x-\mu)}{\sigma}}}{2},\ x > \mu$	μ	$2\sigma^2$	$\frac{e^{\mu t}}{1-\sigma^2 t^2},$ $\lvert t\rvert < \frac{1}{\sigma}$
Log-Normal	$X \sim \text{LogN}(\mu, \sigma^2)$ $\sigma > 0$	$\frac{e^{-\frac{1}{2\sigma^2}[\ln(x)-\mu]^2}}{x\sigma\sqrt{2\pi}}$ $x > 0$	$0,\ x \le 0$ $\int_0^x f(u)\,du,\ x > 0$	$e^{\mu+\frac{1}{2}\sigma^2}$	$e^{2\mu+\sigma^2} \times \left(e^{\sigma^2} - 1\right)$	DNE

NE: Does not exist
NU: Not useful

Continuous Distributions—Continued

Distribution	Notation	pdf: $f(x)$	cdf: $F(x)$	$\mathrm{E}(X)$	$\mathrm{Var}(X)$	MGF: $M_X(t)$
Logistic	$X \sim$ $\text{Logistic}(\mu, \beta)$ $\beta > 0$	$\dfrac{e^{-(x-\mu)/\beta}}{\beta\left[1+e^{-(x-\mu)/\beta}\right]^2}$	$\dfrac{1}{1+e^{-(x-\mu)/\beta}}$	μ	$\dfrac{\pi^2\beta^2}{3}$	$e^{\mu t}\Gamma(1-\beta t)$ $\times\Gamma(1+\beta t)$, $\lvert t\rvert < \dfrac{1}{\beta}$
Normal	$X \sim \mathrm{N}(\mu, \sigma^2)$ $\sigma > 0$	$\dfrac{e^{-\frac{1}{2\sigma^2}(x-\mu)^2}}{\sigma\sqrt{2\pi}}$	$\displaystyle\int_{-\infty}^{x} f(u)\,du$	μ	σ^2	$e^{\mu t+\frac{1}{2}\sigma^2 t^2}$
Pareto	$X \sim$ $\text{Pareto}(\alpha, \beta)$ $\alpha > 0,\ \beta > 0$	$\dfrac{\beta\alpha^\beta}{x^{\beta+1}}$ $x > \alpha$	$0,\ x < \alpha$ $1-\left(\dfrac{\alpha}{x}\right)^\beta,\ x > \alpha$	$\dfrac{\alpha\beta}{\beta-1}$, $\beta > 1$	$\dfrac{\alpha^2\beta}{(\beta-1)^2(\beta-2)}$, $\beta > 2$	DNE
t	$X \sim t_\nu$ $\nu > 0$	$\dfrac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\nu\pi}}$ $\times\left(1+\dfrac{x^2}{\nu}\right)^{-\frac{1}{2}(\nu+1)}$	$0,\ x < 0$ $\displaystyle\int_0^x f(u)\,du,\ x > 0$	0, $\nu > 1$	$\dfrac{\nu}{\nu-2}$, $\nu > 2$	DNE
Uniform	$X \sim \text{Unif}(a, b)$ $b > a$	$\dfrac{1}{b-a}$, $a \le x \le b$	$0,\ x < a$ $\dfrac{x-a}{b-a},\ a \le x \le b$ $1,\ x > b$	$\dfrac{a+b}{2}$	$\dfrac{(b-a)^2}{12}$	$\dfrac{e^{bt}-e^{at}}{t(b-a)}$
Weibull	$X \sim \text{Weib}(\gamma, \beta)$ $\gamma > 0,\ \beta > 0$	$\dfrac{\gamma x^{\gamma-1}e^{-\frac{x^\gamma}{\beta}}}{\beta}$ $x > 0$	$0,\ x \le 0$ $1-e^{-\frac{x^\gamma}{\beta}},\ x > 0$	$\beta^{\frac{1}{\gamma}}\Gamma\left(1+\dfrac{1}{\gamma}\right)$	$\beta^{\frac{2}{\gamma}}k$, $k = \Gamma\left(1+\dfrac{2}{\gamma}\right)$ $-\Gamma^2\left(1+\dfrac{1}{\gamma}\right)$	NU

NE: Does not exist
NU: Not useful