Master's Level Comprehensive Exam - Probability and Mathematical Statistics
Monday August 20, 2012

Note: Start each problem on a new page. You must show work for full credit. The notation describing probability distributions below follows your formula sheet. Be sure to distinguish between, for example, Gam and Gam*.

1. Let P be a probability function and A, B and C be any 3 sets. Prove the following.
   (a) If $A \subset B$ then $P(A) \leq P(B)$. (5 points)
   (b) Suppose that events A, B, and C are independent. Show that A and $(B \cup C)$ are independent events. (5 points)

2. An urn contains two balls numbered 1 and 3. First a ball is drawn randomly from the urn, and then a fair coin is tossed the number of times as the number indicated on the drawn ball. Let X equal the number of heads.
   (a) Find the probability density function for X (10pts).
   (b) Find the mean and variance of X (5 pts).

3. Let $X_1, X_2, \cdots, X_n$ be a sequence of independent and identically distributed Expon(β) random variables.
   (a) Show that $Z = X_1 + X_2 + \cdots + X_n$ has a Gam(n, β) distribution. (5 points)

   Now let $Z = X_1 + X_2 + \cdots + X_N$ with $N \sim$ Geom(p), i.e. Z is a random sum of random variables. We have a hierarchical model with $Z|N \sim$ Gam(N, β) and $N \sim$ Geom(p).

   (b) Find $E(Z)$. (5 points)
   (c) Find Var(Z). (5 points)
   (d) Find the joint probability distribution of (Z, N). (5points)
   (e) Find the marginal distribution of Z. (10 points)

4. Suppose a continuous random variable X has a monotone increasing cumulative distribution function $F_X(x)$. Assume that the density function f_X exists. Verify that the random variable $Y = F_X(X)$ has a uniform distribution over the interval $[0, 1]$. (5 points)

5. Let X be a Poisson random variable with mean λ and let $Y = e^X$. Find the probability distribution of Y. (10 points)

6. Let X and Y have joint density

$$f(x, y) = \begin{cases} \lambda^2 e^{-\lambda(x+y)} & \text{if } 0 < x < \infty, 0 < y < \infty \\ 0 & \text{elsewhere} \end{cases}$$

   (a) Find the joint density of $U = X/Y$ and $V = X + Y$. (10 points)
   (b) Are U and V independent? Justify your answer. (5 points)
   (c) Find the marginal distributions of U and V. (5 points)

7. The discrete random variable X has 4 possible values: 0, 1, 2, and 3. Consider the 3 models given in the accompanying table of probabilities.

	θ_1	θ_2	θ_3
0	.2	.2	.4
1	.5	.6	.3
2	.1	.1	.2
3	.2	.1	.1

A Bayesian statistician places the following prior on $\Theta = (\theta_1, \theta_2, \theta_3)'$: $\pi(\theta_1) = 0.3$, $\pi(\theta_2) = 0.4$, and $\pi(\theta_3) = 0.3$.

(a) Find the unconditional probability that $X = 1$. (5pts)

(b) Find the posterior probability $\pi(\theta_3 | X = 1)$. (5pts)

8. Let X_i be a random variable distributed as $N(i, i^2), i = 1, 2, 3$. Let X_1, X_2, X_3 be mutually independent.

(a) Find a random variable $U = U(X_1, X_2, X_3)$ such that U has a chi-squared distribution with 3 degrees of freedom. (5 points)

(b) Find a random variable $V = V(X_1, X_2, X_3)$ such that V has an F distribution with 1 numerator and 2 denominator degrees of freedom. (5 points)

9. Suppose $X_1, \cdots, X_n$ represent a random sample from a population with pdf

$$f_X(x|\theta) = \frac{2x}{\theta} \exp\left\{-x^2/\theta\right\} I_{(0,\infty)}(x), \quad \theta > 0.$$

Given: $E(X) = (1/2)\sqrt{\pi\theta}$ and $\text{Var}(X) = \theta\left(1 - \frac{\pi}{4}\right)$. The maximum likelihood estimator for θ is

$$\widehat{\theta} = (1/n)\sum_{i=1}^{n} X_i^2$$

(a) Find the maximum likelihood estimator for $E(X)$ and $\text{Var}(X)$. Justify your answer. (6 points)

(b) Is the MLE for θ a Uniform Minimum Variance Unbiased Estimator? Justify your answer. (8pts)

(c) Find an exact $(1 - \alpha)100\%$ confidence interval for θ. (Hint: $2X^2/\theta \sim \chi^2_2$.) (10pts)

10. State whether the statement is ALWAYS, SOMETIMES, or NEVER true. In each case justify your answer.

(a) An unbiased estimator is consistent. (5 points)

(b) A consistent estimator is unbiased. (5 points)

11. Let $X_1, X_2, \cdots, X_n$ be a random sample from $\text{N}(\mu, \theta)$.

(a) Find the Cramer-Rao Lower Bound (CRLB) for all unbiased estimators of θ. (10 points)

(b) We know that the sample variance S^2 is unbiased for θ.

   i. Find its variance. (Hint: Remember the distribution of $(n-1)S^2/\theta$.) (5 points)

   ii. Compare the variance to the CRLB. (3 points)

   iii. Discuss, paying particular attention to what happens with large samples. (5 points)

12. Let $X_1, \cdots, X_n$ be iid Poisson(λ).

   (a) Using the Central Limit Theorem find a large sample level α test of $H_0 : \lambda = \lambda_0$ versus $H_1 : \lambda \neq \lambda_0$. Be sure to identify the test statistic. (10 points)

   (b) Find a large sample level α LRT of $H_0 : \lambda = \lambda_0$ versus $H_1 : \lambda \neq \lambda_0$. Be sure to identify the test statistic. (10 points)

13. Let $X_1, \cdots, X_n$ be a random sample from a Uniform $(\theta, 2\theta)$ with $\theta > 0$.

   (a) Show that the smallest order statistic $X_{(1)}$ and the largest order statistic $X_{(n)}$ are a pair of joint minimal sufficient statistics for θ. (10 points)

   (b) Find the MLE of θ. Is it a function of a sufficient statistic? (10 points)