Comprehensive Exam
Probability and Mathematical Statistics
August 19, 2013

Note: Start each problem on a new page.

1. (5pts) Show that if A and B are independent then A^c and B^c are independent.

2. (7pts) Suppose PH, a calculator company has 3 product inspectors. Let E denote the event that an error occurred and I_i denote inspector i, $i = 1, \cdots, 3$. The probability a defective product slips by an inspector is

$$P(E|I_i) = \begin{cases} 0.05 & I_1 \\ 0.10 & I_2 \text{ and } I_3 \end{cases}$$

Assume that inspectors I_1, I_2 each inspect 40% of the products and and I_3 inspects 20% of the products. Given a defective product slips by, what is the probability that inspector 3 was the culprit? Show all work.

3. (8pts) Let $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_n$ be two independent random samples each of size n from distributions with respective means μ_X and μ_Y and common variance σ^2. Both means and the common variance are finite. What is the limiting distribution of

$$\frac{\sqrt{n}\left[\left(\overline{X}_n - \overline{Y}_n\right) - (\mu_X - \mu_Y)\right]}{\sigma\sqrt{2}}.$$

Be sure to justify your answer.

4. (7pts) Let X and Y be continuous random variables whose expectations exist. Show that for any integrable function g

$$E(g(X)Y) = E(g(X))\,E(Y|X).$$

$$f(x) = (\theta + 1)f_1(x) - \theta f_2(x)$$

for $x > 0$ and $0 \le \theta \le 1$ and $f(x) = 0$ for $x \le 0$.

5. (5pts) Let $X \sim Expon(\theta)$ and $Y \sim Expon(\lambda)$ with X and Y independent. Let $Z = \max(X, Y)$. Find $f_Z(z)$.

6. Let X_1 and X_2 be independent $Expon(1)$ random variables.

   (a) (8pts) Let $U = X_1 - X_2$ and $V = X_1 + X_2$. Find the joint distribution of (U, V).

   (b) (5pts) Are U and V independent? Justify your answer.

7. Consider two random variables X and Y with joint pdf

$$f_{XY}(x, y) = \begin{cases} 1/2; & 0 < y < x < 2 \\ 0; & \text{elsewhere} \end{cases}$$

   (a) (5pts) Find $f_{X|Y}(x|y)$.

(b) (3pts) Find $E(X|Y=y)$.

(c) (5pts) Find $P(X \leq 1|Y=y)$.

8. (6pts) Let $X|Y \sim N(Y,1)$ and $Y \sim N(\mu, \sigma^2)$. Find the $E(X)$ and $Var(X)$.

9. Suppose $X \sim Uniform(0,\theta)$ with $\theta \sim Pareto(\alpha, \beta)$ parameterized as

$$\pi(\theta) = \frac{\beta\alpha^\beta}{\theta^{\beta+1}} I_{(\alpha,\infty)}(\theta).$$

We consider $\alpha > 0$ and $\beta > 0$ to be known constants.

(a) (5pts) Find the posterior density of θ.

(b) (3pts) Find the Bayes Estimator of θ under squared error loss.

10. Suppose $X_i \overset{iid}{\sim} Geom(\theta),\ i = 1, \cdots, n$.

(a) (8pts) Find the MLE of θ.

(b) (15 pts) Give the large sample approximate distribution of the MLE and use it to find an approximate $100(1-\alpha)\%$ confidence interval for θ.

11. Let $X_1, \cdots, X_n$ be a random sample from the density

$$f(x|\theta) = \frac{2x}{\theta^2} I_{(0,\theta)}(x)$$

(a) (5pts) Show that the the MLE of θ is $Y_n = \max(X_1, \cdots, X_n)$.

(b) (10pts) Verify Y_n is a complete sufficient statistic.

(c) (10pts) Is there a UMVUE of θ? If so, find it. If not - justify your answer.

(d) (3pts) Are the theoretical results involving the Cramer-Rao Lower Bound relevant for this problem? Why or why not?

12. (10pts) Suppose $X_1, \cdots, X_n \overset{iid}{\sim} Poisson(\theta_1)$ and $Y_1, \cdots, Y_m \overset{iid}{\sim} Poisson(\theta_2)$. We wish to test $H_0 : \theta_1 = \theta_2 = \theta_0$ versus $H_a : \theta_1 \neq \theta_2$. Find an approximate large-sample level α likelihood ratio test. Do not spend a lot of time trying to simplify the test statistic beyond the obvious.