Probability and Mathematical Statistics Comprehensive Exam
Tuesday August 19, 2014
Show work for full credit. Start each problem on a new page.

1. (5pts) Given that $P(A) > 0$ and $P(B) > 0$ prove or disprove that if $P(A) = P(B)$ then $P(A|B) = P(B|A)$.

2. (10pts) There are 3 coins in a box. One is a two-headed coin, another is a fair coin, and the third is a biased coin that comes up heads 75 percent of the time. When one of the 3 coins is selected at random and flipped, it shows heads. What is the probability it was the two-headed coin?

3. (5pts) If $E(X|Y = y) = c$ for all y show that $Cov(X, Y) = 0$.

4. Suppose that X and Y are jointly distributed random variables with joint pdf

$$f_{XY}(x,y) = \begin{cases} 2; & 0 < x < y < 1 \\ 0: & \text{elsewhere} \end{cases}$$

(10 pts) Let $U = X + Y$. Find the distribution of U.

5. (5pts) A fair coin is tossed 3 times independently. Define random variables X and Y by

$$\begin{aligned} X &= \text{the number of heads observed in the 3 tosses} \\ Y &= \text{the number of tails occurring before the first head} \end{aligned}$$

If $\{TTT\}$ is observed then set $Y = 0$. Are X and Y independent? Justify your answer.

6. Consider a random sample $X_1, \cdots, X_n$ from a distribution with cdf $F_X(x) = 1 - a/x$ if $a \leq x < \infty$ and 0 otherwise where a is a positive constant.

   (a) (5 pts) What is the cdf of the smallest order statistic $Y_n = \min(X_1, \cdots, X_n)$?

   (b) (5 pts) Find the limiting distribution of Y_n.

7. (5pts) Two 6 sided dice, one red and one green, are thrown n times. Let X denote the number of throws in which the number on the red die exceeds the number on the green die. What is the distribution of X? Justify your answer.

8. (5pts) Let $X \sim N(0,1)$. Show that $X^2 \sim \chi^2_1$ using the moment generating function.

9. Suppose $X_1, \cdots, X_n$ represent a random sample from a population with pdf

$$f_X(x|\theta) = \frac{2x}{\theta} \exp\left\{-x^2/\theta\right\} I_{(0,\infty)}(x), \quad \theta > 0.$$

Given: $E[X] = (1/2)\sqrt{\pi\theta}$ and $\text{Var}(X) = \theta(1 - \pi/4)$. Hint 1: Note that $X \sim Weib(2, \theta)$. Hint 2: The following may be useful: $X^2/\theta \sim Gamma(1, 1)$.

(a) (10pts) Find the maximum likelihood estimator (MLE) of θ. In the interest of time you do not need to confirm maximization with a second derivative test.

(b) (10pts) Find the mean and variance of the MLE and show that it is consistent for θ. (Hint 2 may be useful here.)

(c) (10pts) Find the UMVUE for $Var(X)$ and justify why it is UMVUE.

(d) (10pts) Find an exact $(1-\alpha)100\%$ confidence interval for θ (Hint 2 may be useful).

(e) (5pts) Find an approximate large sample level α likelihood ratio test of $H_0 : \theta = \theta_0$ versus $H_a : \theta \neq \theta_0$. (You do not need to spend a lot of time simplifying the LRT statistic,)

(f) (10pts) Find an exact level α likelihood ratio test of $H_0 : \theta = \theta_0$ versus $H_a : \theta \neq \theta_0$. (Hint 2 may be useful).