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The Binomial Theorem

Discrete Mathematics or Introduction to Proof

Elizabeth A. Burroughs, *Montana State University*
Elizabeth G. Arnold, *Colorado State University*
Elizabeth W. Fulton, *Montana State University*
Douglas E. Ensley, *Shippensburg University*
Nancy Ann Neudauer, *Pacific University*

1.1 Overview and Outline of Lesson

The binomial theorem is studied in college-level Discrete Mathematics or Introduction to Proof courses, and its use is included in most high school mathematics standards. This lesson can fit into a course after students have learned that $\binom{n}{k}$ counts the number of ways to select k different objects from a set of n objects. This lesson examines binomial expansions, binomial coefficients, and the appearance of the binomial theorem in secondary mathematics. Undergraduates use combinatorial reasoning to expand the general expression $(x + y)^n$ and apply the binomial theorem in various problems. They also analyze hypothetical student work in order to develop skills in understanding school student thinking about expanding binomials and in creating questions to guide school students' understanding.

Throughout this lesson we use the term *arithmetic triangle* to refer to the triangular array of natural numbers that is also known as Pascal's triangle and Yang Hui's triangle, among other names.

Different courses in Discrete Mathematics or Introduction to Proof may organize the material contained in this lesson in different ways. This lesson proceeds by showing that the number of strings made of k x's and $n - k$ y's is $\binom{n}{k}$. When these strings are interpreted as multiplication of x's and y's, then each string composed of k x's and $n - k$ y's becomes $x^k y^{n-k}$, and the binomial theorem follows. Along the way, the lesson establishes that the entries in the arithmetic triangle can be computed by $\binom{n}{k}$ and that $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

1. Launch—Pre-Activity
   Prior to the lesson, undergraduates complete a Pre-Activity where they identify patterns in the arithmetic triangle and look for patterns among the expansions of $(x + y)^2$, $(x + y)^3$, and $(x + y)^4$. Instructors can launch the lesson by reviewing undergraduates' responses on the Pre-Activity.

2. Explore—Class Activity

   - *Problems 1–4*:
     Undergraduates use combinatorial reasoning to count the number of 4-character strings that use 3 of one

character and 1 of another. They use this reasoning to reconsider the three binomial expansions from the Pre-Activity to identify the total number of terms in each binomial expansion and the number of like terms in each binomial expansion. They reason through the $n = 4$ instance of $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$. Then, undergraduates use combinatorial reasoning to determine the coefficients of terms in the expansion of $(x + y)^5$, examining why the coefficients of terms in binomial expansions are called "binomial coefficients" (i.e., $\binom{n}{k}$).

- *Problems 5 & 6*:
  Undergraduates generalize the expansion of $(x + y)^n$. Problem 5 provides instructors an opportunity to formally state and prove the binomial theorem and to address how and when the binomial theorem appears in secondary mathematics. Undergraduates apply the binomial theorem in Problem 6.
- *Problems 7 & 8*:
  Undergraduates analyze hypothetical student work to make sense of the binomial theorem. Problem 7 highlights two different perspectives of the binomial coefficients in the binomial theorem, and undergraduates will develop questions to guide the hypothetical students' understanding. In Problem 8, undergraduates examine how a student incorrectly applied the binomial theorem to expand $(2x - y)^4$. Undergraduates are prompted to identify correct and incorrect thinking in the student's work and to consider how they would respond to the student to help guide the student's understanding of the binomial theorem.

3. Closure—Wrap-Up
   Conclude the lesson by summarizing the connections between the arithmetic triangle and the binomial theorem and how and when the binomial theorem appears in secondary mathematics. Additionally, discuss how undergraduates used combinatorial reasoning throughout the lesson.

1.2 Alignment with College Curriculum

The binomial theorem is a topic that fits naturally in a Discrete Mathematics or an Introduction to Proof course. The binomial theorem offers undergraduates an opportunity to learn combinatorial proof techniques and contrast them with the algebraic techniques they are often more familiar with.

1.3 Links to School Mathematics

The binomial theorem has direct connections to multiplying binomials, a skill students use throughout their mathematical studies. Prospective teachers should understand the process of multiplying binomials beyond rote procedure and memorization. By studying connections between the arithmetic triangle and the binomial theorem, prospective teachers will develop their abilities to use combinatorial reasoning.

This lesson highlights:

- Connections between the binomial theorem and the arithmetic triangle;
- Using patterns to develop a combinatorial understanding of the arithmetic triangle and the binomial theorem.

This lesson addresses several mathematical knowledge and practice expectations included in high school standards documents, such as the Common Core State Standards for Mathematics (CCSSM, 2010). High school students are expected to know and apply the binomial theorem for binomial expansions and be able to determine binomial coefficients using the arithmetic triangle (c.f. CCSS.MATH.CONTENT.HSA.APR.C.5). This lesson also provides opportunities for prospective teachers to think about the reasoning of others, construct sound mathematical arguments, and look for and make use of structure.

1.4 Lesson Preparation

Prerequisite Knowledge

Undergraduates should know:

- The multiplication and addition rules for solving counting problems;
- That $\binom{n}{k}$ counts the number of ways to select k different objects from a set of n objects.

Learning Objectives

In this lesson, undergraduates will encounter ideas about teaching mathematics, as described in Chapter 1 (see the five types of connections to teaching listed in Table 1.2). In particular, by the end of the lesson undergraduates will be able to:

- Use combinatorial notation to write the expansion of $(x+y)^n$;
- Apply the binomial theorem to expand binomials and to determine specific coefficients of binomial expansions;
- Identify combinatorial patterns and reasoning in both the arithmetic triangle and the proof of binomial theorem;
- Identify connections between the binomial theorem and the mathematics of secondary school;
- Examine hypothetical school student work in order to identify what a school student does and does not yet understand about the binomial theorem and pose questions to help guide school students' understanding about the use of the binomial theorem.

Anticipated Length

Two 50-minute class sessions.

Materials

The following materials are required for this lesson.

- Pre-Activity (assign as homework prior to Class Activity)
- Class Activity (print Problems 1–4, 5–6, and 7–8 to pass out separately)
- Homework Problems (assign at the end of the lesson)
- Assessment Problems (include on quiz or exam after the lesson)

All handouts for this lesson appear at the end of this lesson, and LaTeX files can be downloaded from INSERT URL HERE.

1.5 Instructor Notes and Lesson Annotations

Before the Lesson

Assign the Pre-Activity as homework to complete in preparation for the lesson, and ask undergraduates to bring their solutions to class on the day you start the Class Activity.

Pre-Activity Review (15 minutes)

As a class, review undergraduates' responses to the Pre-Activity.

For Problem 1, first ask undergraduates to share some of the patterns they observed in the arithmetic triangle. They will likely have different levels of familiarity with the triangle, and you can use this discussion to establish a common set of patterns to refer to throughout the lesson.

Pre-Activity Problem 1

1. Below are the first seven rows of the *arithmetic triangle*, also known as Yang Hui's triangle or Pascal's triangle, among other names. By custom, the rows and entries are numbered starting at 0.

$$\begin{array}{ccccccccccccc} & & & & & & 1 & & & & & & \\ & & & & & 1 & & 1 & & & & & \\ & & & & 1 & & 2 & & 1 & & & & \\ & & & 1 & & 3 & & 3 & & 1 & & & \\ & & 1 & & 4 & & 6 & & 4 & & 1 & & \\ & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\ 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \end{array}$$

(a) Write down at least three patterns that you observe.

Sample Responses:

- Each row starts and ends with the number 1.
- It's symmetric. The second and second to last number in a row are the same.
- Every number is the sum of the two above it.
- The second diagonal is the counting numbers.
- The sum of each row is a power of two.

Commentary:

Some undergraduates may already know that $\binom{n}{k}$ is entry k of row n of the arithmetic triangle, but others may not. Problem 3 of the Class Activity and Problem 1 of the Homework will lead undergraduates to establish this pattern in the triangle.

(b) Generate the next row of the triangle using some or all of these patterns.

Solution:

1 7 21 35 35 21 7 1

Commentary:

From our experience, undergraduates will generally use the "Every number is the sum of the two above it" pattern to generate this row.

We have found that many undergraduates are familiar with this triangle from high school and often remember it by the name "Pascal's triangle." Discuss the history of this triangle and how other mathematicians also discovered this triangular array of numbers. This discussion may include the following ideas (see also Ensley & Crawley (2006); Wilson & Watkins (2013)).

- **Mathematicians who have discovered the triangle**
  - The French mathematician/philosopher **Blaise Pascal** (1623–1662) wrote his Traité du triangle arithmétique (Treatise on the Arithmetic Triangle) in 1654, but this familiar triangular array of numbers was far from unknown to the world at this time. Even within Europe, Pascal's treatise was built up over several generations from correspondence and competition, just as mathematics is developed to this day.
  - The Indian mathematician **Halayudha** (ca 975) documents applications of the arithmetic triangle going back as far as the 2nd century in India.
  - In Iran the triangle is referred to as the **Khayyam triangle** after Persian poet Omar Khayyám (1048–1131), though there is evidence of earlier knowledge there, too.
  - The story is similar in the Chinese mathematics tradition, where the triangle is named for **Yang Hui** (1238–1298) even though it was known at least 200 years earlier in China.
- **Why it is also referred to as the "arithmetic triangle" instead of only by a person's name**
  The challenges of ancient communication and the scarcity of existing records guarantee there is no way to identify a single person who discovered this triangular array of numbers, so it is also referred to as "the arithmetic triangle," which is what Pascal called it in the treatise that he did not live to see published.

- **How it arose in different cultures**
  More interesting than conversations about intellectual provenance is the wide variety of reasons the arithmetic triangle arose in these cultures:
  - The early Indian and Persian work was focused on combinatorial aspects, some motivated by patterns in chants and poetry.
  - The Chinese motivation included methods for approximating roots of numbers, related to what we now call the binomial theorem.
  - And Pascal famously corresponded with Pierre de Fermat on problems involving gambling and probability, providing yet another application for the triangle entries.

To conclude the discussion of Problem 1, emphasize the following connection to teaching.

Discuss This Connection to Teaching

Secondary school teachers are called on by many mathematics content standards (e.g., CCSSM) to help students see how patterns in the arithmetic triangle can be used when computing coefficients in a binomial expansion.

Problems 2 and 3 focus undergraduates' attention on a hypothetical student, Anton, who is investigating why the patterns in the arithmetic triangle are the same as the patterns of coefficients in binomial expansions. Anton's work lays the foundation for the combinatorial reasoning on which the Class Activity is built. For the Pre-Activity review, it is sufficient for undergraduates to notice the patterns; they will return to developing explanations in the Class Activity. Give undergraduates a few minutes to review the context of Anton's work.

Context for Pre-Activity Problem 2

In high school, Anton learned that binomial expansions can be computed using the arithmetic triangle, and he wants to investigate why the numerical pattern in the arithmetic triangle emerges from the algebraic process of computing powers of a binomial. As part of his investigation, Anton expanded each of the following expressions using the distributive property of multiplication over addition, without simplifying the expressions using the commutative or associative properties of multiplication. Below are the expansion he computed, and all of his computations are correct.

$$\begin{aligned}(x+y)^2 &= (x+y)(x+y)\\ &= xx + xy + yx + yy\end{aligned}$$

$$\begin{aligned}(x+y)^3 &= (x+y)(x+y)(x+y)\\ &= xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy\end{aligned}$$

$$\begin{aligned}(x+y)^4 &= (x+y)(x+y)(x+y)(x+y)\\ &= xxxx + xxxy + xxyx + xxyy + xyxx + xyxy + xyyx + xyyy + yxxx + yxxy\\ &\quad + yxyx + yxyy + yyxx + yyxy + yyyx + yyyy\end{aligned}$$

Next, Anton looked for patterns in the types of each term, making a table listing the terms with the following attributes.

$(x+y)^2$

Two x's and Zero y's	One x and One y	Zero x's and Two y's
xx	xy yx	yy

$(x+y)^3$

Three x's and Zero y's	Two x's and One y	One x and Two y's	Zero x's and Three y's
xxx	xxy xyx yxx	xyy yxy yyx	yyy

$(x+y)^4$

Four x's and Zero y's	Three x's and One y	Two x's and Two y's	One x and Three y's	Zero x's and Four y's
$xxxx$	$xxxy$ $xxyx$ $xyxx$ $yxxx$	$xxyy$ $xyxy$ $xyyx$ $yxxy$ $yxyx$ $yyxx$	$xyyy$ $yxyy$ $yyxy$ $yyyx$	$yyyy$

Before working on Problem 2, ask undergraduates what method they think Anton used to expand $(x+y)^2$ and if this method works for all polynomial multiplication and discuss the following connection to teaching. Undergraduates will likely say they used "FOIL," which is a commonly-used secondary school mnemonic to remind students that when multiplying binomials they multiply the First, Outside, Inside, and Last pairs of terms.

Discuss This Connection to Teaching

Secondary students typically use the FOIL method to expand binomial products such as $(x+y)^2$. However, they often view the FOIL method as a procedure to memorize and use it for **all** polynomial multiplication, which can lead to computational errors when there are more than four terms in the resulting product. This can interfere with sense-making, because students ought to make sense of the procedure, understand when it can and cannot be used, and understand that the FOIL method is a repeated application of the distributive property. Prospective teachers who also understand these nuances can help their future students.

Pre-Activity Problem 2

2. Anton first looked for patterns in the total number of terms in each of his expansions. He noticed that his expansion of $(x+y)^2$ has 4 terms, his expansion of $(x+y)^3$ has 8 terms, and his expansion of $(x+y)^4$ has 16 terms. Without multiplying everything out and counting, how can you use the pattern Anton noticed to predict the number of terms that $(x+y)^5$ would have?

Sample Responses:

- Anton's number of terms in each expansion is 2^n where n is the value of the exponent. So if Anton expanded $(x+y)^5$, then there would be $2^5 = 32$ terms.
- I notice the pattern is 2^n, where n is the row of the arithmetic triangle, so $(x+y)^5$ would have $2^5 = 32$ terms.

Pre-Activity Problem 3

3. In his tables, Anton observed that the number of expressions of each type correspond to entries of the arithmetic triangle. For example, in his first table for $(x+y)^2$ he counted the following number of terms in each category which corresponded to Row 2 of the arithmetic triangle.

Two x's and Zero y's	One x and One y	Zero x's and Two y's
xx	xy	yy
	yx	
(1)	(2)	(1)

Use the pattern he found to conjecture why it is customary that the rows and entries of the arithmetic triangle are numbered beginning with 0.

Sample Responses:

- Because $(x+y)^0 = 1$ and the only value in the row numbered 0 of the arithmetic triangle is 1.
- Because the pattern is easier to identify if the row of the triangle labeled, for example, "row 4" corresponds to $(x+y)^4$.

Class Activity: Problems 1–4 (20 minutes)

Pass out **Problems 1–4** of the Class Activity.

Instruct undergraduates to work in groups on Problems 1 and 2 and then discuss the solutions, asking undergraduates to share their work when appropriate. See Chapter 1 for guidance on facilitating group work and selecting and sequencing student work for use in whole-class discussion.

Class Activity Problem 1

1. Consider a string of four letters made up of only the letters x or y.

— — — —

(a) How many such strings are there? Explain your reasoning.

Sample Responses:

- $2^4 = 16$, because I have two choices (x or y) for each blank.
- Using the multiplication principle, there are $2 \cdot 2 \cdot 2 \cdot 2 = 16$ distinct strings.

(b) List the strings that contain exactly three x's. How many are there? Explain using combinatorial reasoning.

The strings are

$$xxxy \qquad xxyx \qquad xyxx \qquad yxxx$$

Sample Responses:

- 4, because $\binom{4}{3} = 4$. There are 4 blanks and we are choosing 3 of them to be an x, and the order of the x's doesn't matter.
- 4, because $\binom{4}{1} = 4$. There are 4 blanks and if we choose 1 of them to be a y, the other 3 are x, and the order of the x's doesn't matter.

(c) List the strings that contain exactly one y. How many are there? Explain using combinatorial reasoning.

The strings are

$$xxxy \qquad xxyx \qquad xyxx \qquad yxxx$$

Sample Responses:

- 4, because $\binom{4}{1} = 4$. There are 4 blanks and we are choosing 1 blank to be a y.
- 4, because placing 1 y and then 3 x's is the same as placing 3 x's and then 1 y, and we can choose exactly 3 blanks to place an x with $\binom{4}{3} = 4$.

(d) List the strings that contain exactly three y's. How many are there? Explain using combinatorial reasoning.

The strings are

$$xyyy \qquad yxyy \qquad yyxy \qquad yyyx$$

Sample Responses:

- 4, because $\binom{4}{3} = 4$. There are 4 blanks and we are choosing 3 of them to place a y, and the order of the y's doesn't matter
- $\binom{4}{1} = 4$ or $\binom{4}{3} = 4$, depending on whether I think of it as placing 3 y's and then 1 x or as placing 1 x and then 3 y's.

Commentary:
If undergraduates do not specifically identify the symmety in their answers, facilitate a discussion that highlights that if you choose 1 of the 4 blanks to place an x, then you are also specifying 3 of the 4 blanks to place a y, for example.

For Problems 2 & 3, refer to Anton's table from the Pre-Activity.

Class Activity Problem 2

2. Reexamine the table Anton created in his quest to understand binomial patterns. Notice the terms in his expansion of $(x + y)^4$ with three x's and one y are: $xxxy, xxyx, xyxx$, and $yxxx$. Also notice that using the context of Problem 1, Anton has listed all of the strings with three x's and one y.

(a) Explain to Anton why $\binom{4}{1}$ counts the number of such terms.

Sample Response:

These four are all of the possible strings that have exactly 1 y and 3 x's. This is what $\binom{4}{1}$ gives: how to place 1 character in a 4-character string and filling in the rest of the string with x's.

(b) Now, use associative and commutative properties of multiplication to generate like terms. Explain to Anton why the coefficient of x^3y in the expansion of $(x+y)^4$ is exactly the same as the number of terms that have three x's and one y.

Sample Response:

Each of these terms is equivalent to x^3y. Since there are four such terms, each occuring once, when you combine like terms, you get the coefficient of x^3y, which is 4, and logically, is $\binom{4}{1}$.

Problem 3 examines the $n = 4$ instance of $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

Class Activity Problem 3

3. Look again at Anton's table for the expansion of $(x+y)^4$. Note that the $(x+y)^4$ entries in the column labeled "Two x's and Two y's" are of two types: those that start with x and those that start with y. Explain how to generate each of these entries by starting with appropriate entries in the table for $(x+y)^3$.

Solution:

Those that start with x can be constructed by prepending an x on the $(x+y)^3$ entries in the column labeled "One x and Two y's," and those that start with y can be constructed by prepending a y on the $(x+y)^3$ entries in the column labeled "Two x's and One y."

Commentary:

Note that this explains the relation $\binom{4}{2} = \binom{3}{1} + \binom{3}{2}$ without the need to compute the values involved. This is the pattern used to construct the arithmetic triangle, providing evidence for the observation that the counting numbers $\binom{n}{k}$ are the values in the triangle. You can direct undergraduates to use similar reasoning to predict how many terms will be in a hypothetical $(x+y)^5$ table under the column heading, "Two x's and Three y's" by referencing Anton's table for $(x+y)^4$.

In Problem 4, undergraduates apply combinatorial reasoning to an expansion that is not yet in Anton's chart, moving toward the expansion of the general case $(x+y)^n$.

Class Activity Problem 4

4. Suppose Anton were to expand

$$(x+y)^5 = (x+y)(x+y)(x+y)(x+y)(x+y)$$

From his earlier work, he knows that he will have 5 terms in the expansion, one each corresponding to x^5, x^4y, x^3y^2, x^2y^3, xy^4, and y^5.

Determine the coefficients in the expansion, and explain to Anton how you determined the coefficients.

Sample Responses:

I used combinations. $\binom{5}{5} = 1$, $\binom{5}{4} = 5$, $\binom{5}{3} = 10$, $\binom{5}{2} = 10$, $\binom{5}{1} = 5$, and $\binom{5}{0} = 1$. We are choosing how to arrange strings of length five and specifying that all 5 are x's, then that 4 are x's, then 3, 2, 1, and 0.

Commentary:

Undergraduates will usually write $\binom{5}{5} = 1$ instead of $\binom{5}{0} = 1$ as the first coefficient. Asking them to explain what that combination represents in the context of the problem will highlight whether they are choosing the characters in the string to be x's or choosing them to be y's. The discussion of symmetry from Problem 1 is useful for moving from undergraduates' observations about the coefficients based on counting arguments to the statement of the binomial theorem as it is usually written.

Advice on Teaching the Lesson over Two Days

If you are teaching the lesson over two class periods, this can be an effective place to stop for the day. At this point in the lesson, you can demonstrate that the arithmetic triangle can be rewritten using combinatorics, as follows, emphasizing that the triangle can be thought of as displaying the results of computing combinations, and undergraduates can complete Problem 1 from the homework, which provides a generalization of the observation from Problem 3 in the Class Activity.

$$\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}
\end{array}$$

See Chapter 1 for guidance on using **exit tickets** to facilitate instruction in a two-day lesson.

Class Activity: Problems 5–6 (25 minutes)

If you assigned Problem 1 from the homework, you can discuss that result to initiate the discussion.

Distribute **Problems 5 and 6** of the Class Activity. We suggest you discuss the solution to Problem 5, then formally state the binomial theorem, and conclude by discussing the solutions to Problem 6.

Class Activity Problem 5

5. **Expanding Binomial Products.**
Use combinatorial notation to write the expansion of $(x+y)^n$. Explain how you determined this is the appropriate expression.

Sample Response:

$$(x+y)^n = \binom{n}{n}x^n y^0 + \binom{n}{n-1}x^{n-1}y^1 + \cdots + \binom{n}{1}x^1 y^{n-1} + \binom{n}{0}x^0 y^n$$

The number of strings made up of k x's and $n-k$ y's is $\binom{n}{k}$.

Commentary:
Undergraduates will usually write the formula as shown above, where the first coefficient is $\binom{n}{n}$ rather than $\binom{n}{0}$. Prompt undergraduates to notice the alternative

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \cdots + \binom{n}{n-1}x^1 y^{n-1} + \binom{n}{n}x^0 y^n,$$

focusing on symmetry in the arithmetic triangle and the equivalence of the algebraic expressions for $\binom{n}{n}$ and $\binom{n}{0}$.

State the binomial theorem and point out that $\binom{n}{k}$ is called a "binomial coefficient." If you usually ask your students to prove the binomial theorem, assign them to groups to generate an outline of a proof of the binomial theorem based on the combinatorial arguments they have developed thus far. Alternatively, if you prefer to provide a proof in class, we have found that the Class Activity has laid the foundation for undergraduates to understand an instructor-led proof based on combinatorial arguments.

Theorem 1 (Binomial Theorem)**.** *The expansion of the binomial $(x+y)^n$ to an integer power $n \geq 1$ is given by*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Outline of proof. Consider the expansion of $(x+y)^n$, which will consist of a sum of terms of the form $x^{n-k}y^k$, for each integer $0 \leq k \leq n$. For each k, the term $x^{n-k}y^k$ appears in the sum the same number of times as there are strings of length n with $(n-k)x$'s and k y's, which is $\binom{n}{k}$. Combining like terms yields

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

□

Discuss the following connection to teaching, which emphasizes how the binomial theorem is used in secondary mathematics.

Discuss This Connection to Teaching

Expanding binomial products is fundamental to school mathematics, and the binomial theorem is typically taught in Intermediate Algebra as a core content standard. School students often use this theorem to expand binomials to a power higher than 2. Many school students memorize the binomial theorem and rely on the arithmetic triangle to derive the necessary coefficients. Combinatorial reasoning helps undergraduates to understand and express the algebraic patterns and coefficients of the binomial expansions. All undergraduates should understand the combinatorial reasoning that underlies the binomial theorem in order to develop a thorough understanding of binomial expansion.

Problem 6 contains two parts that are representative of the kind of problems found in both high school and undergraduate textbooks that ask students to apply the binomial theorem. We have found that undergraduates will more often use an algebraic approach on Problem 6(a) (finding the coefficient of the term that contains x^7), unless they are prompted to use a combinatorial approach.

Class Activity Problem 6

6. **Applying the Binomial Theorem.**

   (a) What is the coefficient of the term that contains x^7 in the expansion of $(x+4y)^{10}$? Explain how you determined this.

   Sample Responses:

   - Using the binomial theorem, the term that contains x^7 in it is $\binom{10}{7}(x)^7(4y)^3$. Thus, the coefficient is $\binom{10}{7}(1)^7(4)^3 = (120)(1)(64) = 7,680$
   - We need exactly 7 of the 10 binomials to contribute an x and this can be done in $\binom{10}{7} = 120$ ways. Then because the coefficient of the x is 1, we'd have $(1)^7 = 1$. If the exponent of the x is 7, then the exponent of the y must be $10 - 7 = 3$ and since there is a 4 in front of the y, we'd also have $4^3 = 64$ as part of the coefficient. All together, the coefficient is $(120)(1)(64) = 7680$.

(b) Use the binomial theorem to expand $(3x-2y)^5$. Show all of your work.

Solution:

$$\begin{aligned}(3x-2y)^5 &= \binom{5}{5}(3x)^5(-2y)^0+\binom{5}{4}(3x)^4(-2y)^1+\binom{5}{3}(3x)^3(-2y)^2+\binom{5}{2}(3x)^2(-2y)^3\\&\quad+\binom{5}{1}(3x)^1(-2y)^4+\binom{5}{0}(3x)^0(-2y)^5\\&=(1)(243x^5)(1)+(5)(81x^4)(-2y)+(10)(27x^3)(4y^2)+(10)(9x^2)(-8y^3)\\&\quad+(5)(3x)(16y^4)+(1)(1)(-32y^5)\\&=243x^5-810x^4y+1080x^3y^2-720x^2y^3+240xy^4-32y^5\end{aligned}$$

Class Activity: Problems 7 & 8 (15 minutes)

Pass out **Problems 7 and 8** of the Class Activity. Before instructing undergraduates to work on these problems, discuss the following connection to teaching.

Discuss This Connection to Teaching

Problems 7 and 8 focus on analyzing hypothetical students' thinking in order to develop undergraduates' skills in understanding school student thinking and developing questions to guide school students' understanding. All undergraduates (especially prospective teachers) should explore how others use, reason with, and communicate mathematics. These problems also give prospective teachers (and tutors and future graduate students) an opportunity to think about how they would respond to student work in ways that nurture students' assets and understanding and in ways that help develop a students' mathematical understanding.

Problem 7 emphasizes two different perspectives that depend on a specific frame of reference—choosing blanks to place an x or choosing blanks to place a y. The goal of this problem is to help undergraduates see how multiple perspectives can arise in student work. (See Benson et al., 2005, p. 182, for more examples of problems about combinatorial thinking based on hypothetical student work.)

Class Activity Problem 7

7. Evelyn and Ivy were working on Problem 4 in the Class Activity where they determined the coefficients in the expansion of $(x+y)^5$. Evelyn says that the coefficient of x^3y^2 is $\binom{5}{3}=10$ because she was counting the ways to place 3 x's in a five-character string. Ivy claims that the coefficient is $\binom{5}{2}=10$ since she was counting ways to place 2 y's in a five-character string. Write two questions you could ask Evelyn and Ivy to help them find the common ground between their distinct approaches. Explain how your questions might help them.

   Sample Responses:

   - Is there a difference between counting the ways to place 3 x's versus counting the ways to place 2 y's? Why do both $\binom{5}{3}$ and $\binom{5}{2}$ equal 10? What is the coefficient of x^2y^3? These questions will help them see both are correct and you could view it as counting the ways to place x's or counting the ways to place y's, as long as we are specific in what we are counting. It also helps them to recognize the symmetry in the coefficients of their terms.
   - What variable do you want in the strings you are choosing? How could choosing 3 ways to place an x be similar to choosing 2 ways to place a y? The idea of these questions would be to get them to see that if you choose 3 ways to place an x, then the other blanks have to be y's.

- If you count the strings with three x's how many places in the string would we need to fill with y's? If you count the strings with two y's, how many places in the string would we need to fill with x's? These questions will help show that both answers are correct from different perspectives and all elements of the string are still filled.

Commentary:
Posing questions to a hypothetical student may be new and challenging for your undergraduates. If they are stuck, it may be helpful to ask them to first discuss what Evelyn and Ivy understand based on their work. You may also need to remind undergraduates to explain how their questions might help guide Evelyn's and Ivy's understanding.

Problem 8 contains a hypothetical student, Henry, who makes a mistake when applying the binomial theorem. Emphasize the purpose of Problem 8, as follows:

- We provide this example of student work because it illustrates a common error students make when applying the binomial theorem.
- As mathematicians, it is useful to examine someone else's mathematical thinking and to be able to explain any errors that occur. This process helps deepen our own mathematical understanding.
- Prospective teachers need to examine student thinking and reflect on how they can respectfully resolve mathematical disputes because these are skills they will use in their future careers.

Class Activity Problem 8 : Parts a & b

8. Henry, a high school student, expanded $(2x-y)^4$ using the binomial theorem and made some errors. Below is his work.

$$(2x-y)^4 = 2x^4 - 8x^3y - 12x^2y^2 - 8xy^3 - y^4$$

(a) What does Henry understand about the binomial theorem?

Sample Responses:

- He understands the pattern of the exponents—that they decrease/increase by 1.
- He has the correct exponents on the variables.
- He knows the coefficients from Row 4 of the arithmetic triangle are 1, 4, 6, 4, 1, and then he multiplies the coefficient by 2 when the term includes an x and also multiplies the coefficient by -1 when the term includes a y.

(b) What does Henry not yet understand about the binomial theorem?

Sample Responses:

- Henry wasn't consistent in including the 2 with the x and the -1 with the y when taking powers. He multiplied 2 by each of the coefficients in the arithmetic triangle. He forgets to apply the power to the 2 and -1.
- He doesn't yet understand that the coefficients in his final answer should alternate between positive and negative.
- Henry doesn't understand that when he raises something to an even power the resulting coefficients will always be positive, not negative.

Commentary:
Henry's work demonstrates two common errors many students make when using the binomial theorem (incorrectly applying powers to negative signs and applying powers only to the variables in a term and not to the coefficients of the term as well). The correct answer is $16x^4-32x^3y+24x^2y^2-8xy^3+y^4$, and we have found that most undergraduates will recognize that the errors revolve around the coefficients. Undergraduates may also use the term "coefficient" to mean the binomial coefficients $(1,4,6,4,1)$ as well as the coefficients in Henry's solution $(2,-8,-12,-8,-1)$, without distinguishing between the

two. They may say things like "He got the coefficients wrong." and "He understands how to get the coefficients from the arithmetic triangle." If this occurs, ask undergraduates to clarify which coefficients they are referring to.

Problem 8(c) asks undergraduates to evaluate a set of pre-written questions one might ask Henry about his work. Asking undergraduates to evaluate a set of questions is a way to scaffold their understanding of guiding school students' understanding and to help them develop their skills of writing questions on their own.

Class Activity Problem 8 : Part c

(c) Consider the following questions that someone might ask Henry about his work.

i. Explain how the following question could help Henry to advance in his understanding of the binomial theorem:

How is $(2x - y)^4$ *similar to* $(x + y)^4$ *and how is it different?*

Sample Responses:

- This will help Henry realize that $-y$ is really $+(-y)$.
- This can help Henry realize that **all** of $2x$ (both the 2 and x) need to be raised to powers in the binomial theorem, not just the x.

ii. Explain how the following question can help you assess what Henry understands about the binomial theorem:

Why doesn't $(-3y)^2 = -3y^2$*?*

Sample Responses:

- Henry's seems to be incorrectly applying the powers in the binomial theorem. It seems like he may have written $-2x^3$ instead of $(-2x)^3$ and that is how he got his error. Asking this question will help us know if he wrote out the binomial theorem expansion without the parentheses or if he doesn't know how to distribute the power to both the constant and variable.
- This question helps us understand if Henry's error is in using the binomial theorem or if it's a computation error.

iii. Explain why the following question would not help Henry:

Do the exponents and the coefficients look right?

Sample Responses:

- Henry probably thinks his answer is right, so asking this question isn't helpful.
- This would be like saying "your error is in the coefficients" and it is kind of telling Henry where the problem is instead of guiding him to figure out where his error is on his own.
- Henry's exponents in his final answer are correct so this doesn't help him understand that he is not applying the exponents correctly to the coefficients.

Wrap-Up (5 minutes)

Conclude the lesson by discussing the connections between the arithmetic triangle and the binomial theorem, emphasizing how undergraduates have used combinatorial reasoning throughout the lesson. This may include the following ideas:

- As high school students, many of the undergraduates were taught that the binomial theorem was related to the arithmetic triangle, but this relationship may not have been thoroughly explained.

- The purpose of this lesson is to use combinatorial reasoning to explain the connection between the arithmetic triangle and the binomial theorem.
- Using combinatorial reasoning is central to understanding the arithmetic triangle. For the prospective teachers in the class—high school students would not be asked to prove the binomial theorem, but prospective teachers should understand this connection and be able to show this.

You can ask undergraduates to complete an exit ticket, if you choose. See Chapter 1 for guidance on using exit tickets.

Homework Problems

At the end of the lesson, assign the following homework problems. Assign any additional homework problems at your discretion.

Homework Problem 1

1. Generalize the example from Problem 3 of the Class Activity to fill in the proof sketch for the following statement.

   Proposition. The counting numbers $\binom{n}{k}$ satisfy the arithmetic triangle pattern

   $$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

   for all $n \geq k \geq 1$.

   Proof sketch. Each of the $\binom{n}{k}$ arrangements of k x's and $n-k$ y's has one of the following forms:

   (i) An x followed by an arrangement of $k-1$ x's and $n-k$ y's; or
   (ii) A y followed by an arrangement of k x's and $n-k-1$ y's

   The number of arrangements of type (i) is ______ and the number of arrangements of type (ii) is ______, so . . .

   Solution:

   Proof. Each of the $\binom{n}{k}$ arrangements of k x's and $n-k$ y's has one of the following forms:

   (i) An x followed by an arrangement of $k-1$ x's and $n-k$ y's; or
   (ii) A y followed by an arrangement of k x's and $n-k-1$ y's

   The number of arrangements of type (i) is $\binom{n-1}{k-1}$ and the number of arrangements of type (ii) is $\binom{n-1}{k}$, so the total number of arrangements is

   $$\binom{n-1}{k} + \binom{n-1}{k-1},$$

Problem 2 addresses the commonly-held algebraic misconception that $(x+y)^2 = x^2 + y^2$. High school students will have likely seen area models that can be used to illustrate why this is not the case. Having undergraduates also connect their investigations with the binomial theorem to expanding binomials provides another approach to address the conception many school students have when they use this flawed equation.

Homework Problem 2

2. Students with a not-yet-complete understanding of high school algebra commonly argue that $(x+y)^2 = x^2+y^2$. Below are three different approaches a teacher can use to help students see that this is not true.

- One approach is to choose particular values for a and b, say $a = 1$ and $b = 2$, and ask students to plug those values into the left-hand side of the equation and simplify and then plug those same values into the right-hand side of the equation and simplify. Students can compare their answers to see that they are not equal. This approach can quickly show students that $(x + y)^2 \neq x^2 + y^2$ but doesn't give them insight into why the equality does not hold.
- Another approach is to create an area model, such as the one below, which shows students that $(x + y)^2 \neq x^2 + y^2$. In this case the binomial $x + y$ is represented as a segment of length $x + y$, and the product $(x + y)^2$ is represented as the area of a rectangle whose side lengths are both $x + y$. This representation shows students which mathematical components they are missing in their answer, specifically the term $2xy$.

	x	y
x	x^2	xy
y	yx	y^2

- A third approach is to apply the binomial theorem to $(x + y)^2$.

(a) Use the binomial theorem to explain to a high school student why $(x + y)^2 \neq x^2 + y^2$.

Solution:

The binomial theorem provides a formula that can be used to expand a binomial to the second power. The expansion will contain three terms, an x^2 term, a y^2 term, and an xy term, as follows. It is because of the xy term that $(x + y)^2$ is not equal to $x^2 + y^2$.

$$\begin{aligned}(x+y)^2 &= \binom{2}{2}x^2y^0 + \binom{2}{1}x^1y^1 + \binom{2}{0}x^0y^2\\ &= (1)(x^2)(1) + (2)(x)(y) + (1)(1)(y^2)\\ &= x^2 + 2xy + y^2\\ &\neq x^2 + y^2\end{aligned}$$

(b) Compare the "area model" approach with the "binomial theorem" approach. For what kinds of problems would each approach work? What insight does each approach highlight that will help students understand why $(x + y)^2 \neq x^2 + y^2$?

Sample Responses:

- Both the "area model" and the "binomial theorem" approach can be used when expanding binomial products. The area model works when multiplying two binomials, so for example computing $(x+y)^3$ would require two separate uses of the area model. The binomial theorem can be used to expand a binomial to any power.
- While the binomials being multiplied together need to be the same to use the binomial theorem, the area model can multiply two different binomials together, such as $(x + y)(2x - 1)$.

- Each approach highlights the $2xy$ term that is missing when students think $(x+y)^2 = x^2+y^2$. In the area model approach, students can see how each piece of the binomial contributes an x and a y and where the two xy pieces come from.

Problem 3 provides undergraduates an opportunity to examine hypothetical student work and write questions that will help guide the student's mathematical understanding.

Homework Problem 3

3. Charlie incorrectly expanded $(x + y)^2$ as follows.

$$\begin{aligned}(x+y)^2 &= 1x^2y^2 + 2x^1y^1 + 1x^0y^0 \\ &= x^2y^2 + 2xy + 1\end{aligned}$$

(a) What does Charlie understand about the binomial theorem?

Sample Responses:

- Charlie understands the binomial coefficients.
- They correctly understand the coefficients of the x's and that they decrease by 1 each time.

(b) What does Charlie not yet understand about the binomial theorem?

Sample Responses:

- They don't yet understand how the exponents in the binomial theorem work. They give both x and y the same power. However, the power of x plus the power of y should add to 2 for every term. Instead of having 2 x's and no y's for the first term they have 2 of each. The exponents on their second term are correct but their reasoning for it may not be correct. For their last term they have 0 x's and 0 y's but should have 0 x's and 2 y's.
- Charlie doesn't yet understand the pattern in the exponents. They think that exponents count down from n at the same time on both variables.

(c) Write at least two questions that will help Charlie revise their work and develop a deeper understanding of the binomial theorem. Explain how those questions will help Charlie.

Sample Responses:

- Use the distributive property to multiply $(x + y)(x + y)$. How does that answer compare to your original answer? Do we have terms where there are no x's and no y's? These questions will help Charlie recognize where the errors in their work are.
- Plug in $x = 3$ and $y = 4$ into both sides of your equation. What do you notice and what does this tell you? Hopefully Charlie will get $49 = 169$ and realize that their expansion is not correct.
- Can you describe how you got 0 as an exponent in your last term? This will help me understand how Charlie is applying the binomial theorem.

Assessment Problems

The following two problems address ideas explored in the lesson, with a focus on connections to teaching and mathematical content. You can include these problems as part of your usual course quizzes or exams.

Problem 1 assesses undergraduates' use of combinatorial reasoning to explain how to determine a coefficient of a particular term in a binomial expansion.

Assessment Problem 1

1. Consider the expansion of $(x+y)^{10}$.

   (a) Beyond "because the binomial theorem says so," explain why the coefficient of the x^4y^6 term is $\binom{10}{6}$.

   Solution:

   There are two separate points to be made. Putting A and B together leads to a complete explanation.

   A. When there are 4 x's and 6 y's available, there are $\binom{10}{6}$ ways they can be arranged in a line. This is because of the 10 spots in the line available, you choose which 6 spots should get a y (and then the x's must go everywhere else without any more choices being made).

   B. The distributive property tells us that when multiplying out $(x+y)^{10} = (x+y)(x+y)(x+y)\ldots(x+y)$, we choose either an x or a y from each of the 10 binomial terms. Those choices that result in 4 x's and 6 y's will each contribute one to the coefficient of x^4y^6 in the expansion.

   (b) One student states that the coefficient of the x^4y^6 term is $\binom{10}{4}$, and a second student states it is $\binom{10}{6}$. Beyond "because of symmetry," explain why each of them correctly computes the coefficient of x^4y^6.

   Solution:

   - In part A of the explanation above, we could have taken the position that of the 10 spots in the line available, you choose which 4 spots should get an x (and then the y's must go everywhere else without any more choices being made).
   - If you have 10 things and each has either x or y written on it and you know that only 4 of them have x written on them, then the other 6 must have y written on them as those were the only two options. If you instead knew that only 6 had a y written on them, you would also know that 4 had an x written on them as it was the only other option. Either way you know the same thing: 4 have x and 6 have y.

Problem 2 assesses how undergraduates analyze hypothetical student work and how they can write questions to guide the student's understanding.

Assessment Problem 2

2. Tencha, a high school student, expanded $(x-3y)^4$ using the binomial theorem and made some errors. Below is their work.

$$(x-3y)^4 = x^4 - 12x^3y - 18x^2y^2 - 12xy^3 - 3y^4$$

   (a) What does Tencha understand about the binomial theorem?

   Sample Responses:

   - Tencha knows the statement of the theorem exactly right, and they will get the correct answer for any problem about $(x+y)^n$, like $(x+y)^3$.
   - That each term has exponents that add up to 4.

   (b) What does Tencha not yet understand about the binomial theorem?

   Sample Response:

   The correct work is

$$x^4 + 4x^3(-3y)^1 + 6x^2(-3y)^2 + 4x^1(-3y)^3 + (-3y)^4$$

We can see by comparison that Tencha doesn't yet understand that there need to be parentheses around the $-3y$ term when using the binomial theorem.

(c) Write two questions you can ask Tencha to help them revise their work. Explain how your questions could help guide their mathematical understanding.

Sample Responses:

- I would ask Tencha if $2x^2 = (2x)^2$. If Tencha can understand why $2x^2 \neq (2x)^2$, then they may be able to see why $-3y^4 \neq (-3y)^4$.
- Does the left hand side of your equation equal the right hand side of your equation when you plug a value in for x and y and simplify both sides of the equation? This will help them see that there is an error in their work somewhere.
- What happens when we multiply two negative numbers? How does this relate to your expansion? This can help Tencha remember that $(-)(-)$ becomes positive and that when we expand we are multiplying each term so some of the terms in their answer should be positive.

1.6 References

[1] Benson, S., Addingon, S., Arshavsky, N, Cuoco, A., Goldenberg, E. & Karnowski, A. (2005) *Ways to think about mathematics: Activities and investigations for grade 6–12 teachers.* Corwin Press.

[2] Ensley, D. E. & Crawley, J. W. (2006). *Introduction to discrete mathematics: Mathematical reasoning with puzzles, patterns, and games.* John Wiley and Sons.

[3] Epp, S. S. (2011). *Discrete mathematics: An introduction to mathematical reasoning* (Brief Edition). Brooks/Cole Publishing Co.

[4] National Governors Association Center for Best Practices & Council of Chief State School Officers (2010). *Common Core State Standards for Mathematics.* Authors. Retrieved from `http://www.corestandards.org/`

[5] Wilson, R., & Watkins, J. J. (Eds.). (2013). *Combinatorics: Ancient and modern*. (Especially Chapter 7, "The arithmetical triangle" by A. W. F. Edwards). Oxford University Press.

1.7 Lesson Handouts

Handouts for use during instruction are included on the pages that follow. LaTeX files for these handouts can be downloaded from INSERT URL HERE.

NAME: **PRE-ACTIVITY:** BINOMIAL THEOREM (page 1 of 3)

1. Below are the first seven rows of the *arithmetic triangle*, also known as Yang Hui's triangle or Pascal's triangle, among other names. By custom, the rows and entries are numbered starting at 0.

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

(a) Write down at least three patterns that you observe.

(b) Generate the next row of the triangle using some or all of these patterns.

PRE-ACTIVITY: BINOMIAL THEOREM (page 2 of 3)

In high school, Anton learned that binomial expansions can be computed using the arithmetic triangle, and he wants to investigate why the numerical pattern in the arithmetic triangle emerges from the algebraic process of computing powers of a binomial. As part of his investigation, Anton expanded each of the following expressions using the distributive property of multiplication over addition, without simplifying the expressions using the commutative or associative properties of multiplication. Below are the expansion he computed, and all of his computations are correct.

$$\begin{aligned}(x+y)^2 &= (x+y)(x+y)\\ &= xx+xy+yx+yy\end{aligned}$$

$$\begin{aligned}(x+y)^3 &= (x+y)(x+y)(x+y)\\ &= xxx+xxy+xyx+xyy+yxx+yxy+yyx+yyy\end{aligned}$$

$$\begin{aligned}(x+y)^4 &= (x+y)(x+y)(x+y)(x+y)\\ &= xxxx+xxxy+xxyx+xxyy+xyxx+xyxy+xyyx+xyyy+yxxx+yxxy\\ &\quad +yxyx+yxyy+yyxx+yyxy+yyyx+yyyy\end{aligned}$$

Next, Anton looked for patterns in the types of each term, making a table listing the terms with the following attributes.

$(x+y)^2$

Two x's and Zero y's	One x and One y	Zero x's and Two y's
xx	xy	yy
	yx	

$(x+y)^3$

Three x's and Zero y's	Two x's and One y	One x and Two y's	Zero x's and Three y's
xxx	xxy	xyy	yyy
	xyx	yxy	
	yxx	yyx	

$(x+y)^4$

Four x's and Zero y's	Three x's and One y	Two x's and Two y's	One x and Three y's	Zero x's and Four y's
$xxxx$	$xxxy$	$xxyy$	$xyyy$	$yyyy$
	$xxyx$	$xyxy$	$yxyy$	
	$xyxx$	$xyyx$	$yyxy$	
	$yxxx$	$yxxy$	$yyyx$	
		$yxyx$		
		$yyxx$		

2. Anton first looked for patterns in the total number of terms in each of his expansions. He noticed that his expansion of $(x+y)^2$ has 4 terms, his expansion of $(x+y)^3$ has 8 terms, and his expansion of $(x+y)^4$ has 16 terms. Without multiplying everything out and counting, how can you use the pattern Anton noticed to predict the number of terms that $(x+y)^5$ would have?

3. In his tables, Anton observed that the number of expressions of each type correspond to entries of the arithmetic triangle. For example, in his first table for $(x+y)^2$ he counted the following number of terms in each category which corresponded to Row 2 of the arithmetic triangle.

Two x's and Zero y's	One x and One y	Zero x's and Two y's
xx	xy	yy
	yx	
(1)	(2)	(1)

Use the pattern he found to conjecture why it is customary that the rows and entries of the arithmetic triangle are numbered beginning with 0.

NAME: **CLASS ACTIVITY:** BINOMIAL THEOREM (page 1 of 5)

1. Consider a string of four letters made up of only the letters x or y.

   — — — —

   (a) How many such strings are there? Explain your reasoning.

   (b) List the strings that contain exactly three x's. How many are there? Explain using combinatorial reasoning.

   (c) List the strings that contain exactly one y. How many are there? Explain using combinatorial reasoning.

   (d) List the strings that contain exactly three y's. How many are there? Explain using combinatorial reasoning.

2. Reexamine the table Anton created in his quest to understand binomial patterns. Notice the terms in his expansion of $(x+y)^4$ with three x's and one y are: $xxxy, xxyx, xyxx$, and $yxxx$. Also notice that using the context of Problem 1, Anton has listed all of the strings with three x's and one y.

   (a) Explain to Anton why $\binom{4}{1}$ counts the number of such terms.

   (b) Now, use associative and commutative properties of multiplication to generate like terms. Explain to Anton why the coefficient of x^3y in the expansion of $(x+y)^4$ is exactly the same as the number of terms that have three x's and one y.

3. Look again at Anton's table for the expansion of $(x+y)^4$. Note that the $(x+y)^4$ entries in the column labeled "Two x's and Two y's" are of two types: those that start with x and those that start with y. Explain how to generate each of these entries by starting with appropriate entries in the table for $(x+y)^3$.

4. Suppose Anton were to expand

$$(x+y)^5 = (x+y)(x+y)(x+y)(x+y)(x+y)$$

   From his earlier work, he knows that he will have 5 terms in the expansion, one each corresponding to x^5, x^4y, x^3y^2, x^2y^3, xy^4, and y^5.

   Determine the coefficients in the expansion, and explain to Anton how you determined the coefficients.

5. **Expanding Binomial Products.**
   Use combinatorial notation to write the expansion of $(x+y)^n$. Explain how you determined this is the appropriate expression.

6. **Applying the Binomial Theorem.**
   (a) What is the coefficient of the term that contains x^7 in the expansion of $(x+4y)^{10}$? Explain how you determined this.

   (b) Use the binomial theorem to expand $(3x-2y)^5$. Show all of your work.

7. Evelyn and Ivy were working on Problem 4 in the Class Activity where they determined the coefficients in the expansion of $(x + y)^5$. Evelyn says that the coefficient of x^3y^2 is $\binom{5}{3} = 10$ because she was counting the ways to place 3 x's in a five-character string. Ivy claims that the coefficient is $\binom{5}{2} = 10$ since she was counting ways to place 2 y's in a five-character string. Write two questions you could ask Evelyn and Ivy to help them find the common ground between their distinct approaches. Explain how your questions might help them.

8. Henry, a high school student, expanded $(2x - y)^4$ using the binomial theorem and made some errors. Below is his work.

$$(2x - y)^4 = 2x^4 - 8x^3y - 12x^2y^2 - 8xy^3 - y^4$$

   (a) What does Henry understand about the binomial theorem?

   (b) What does Henry not yet understand about binomial theorem?

(c) Consider the following questions that someone might ask Henry about his work.

i. Explain how the following question could help Henry to advance in his understanding of the binomial theorem:

How is $(2x - y)^4$ *similar to* $(x + y)^4$ *and how is it different?*

ii. Explain how the following question can help you assess what Henry understands about the binomial theorem:

Why doesn't $(-3y)^2 = -3y^2$*?*

iii. Explain why the following question would not help Henry:

Do the exponents and the coefficients look right?

NAME: **HOMEWORK PROBLEMS:** BINOMIAL THEOREM (page 1 of 1)

1. Generalize the example from Problem 3 of the Class Activity to fill in the proof sketch for the following statement.

   Proposition. The counting numbers $\binom{n}{k}$ satisfy the arithmetic triangle pattern

   $$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

   for all $n \geq k \geq 1$.

   Proof sketch. Each of the $\binom{n}{k}$ arrangements of k x's and $n-k$ y's has one of the following forms:

   (i) An x followed by an arrangement of $k-1$ x's and $n-k$ y's; or

   (ii) A y followed by an arrangement of k x's and $n-k-1$ y's

   The number of arrangements of type (i) is ______ and the number of arrangements of type (ii) is ______, so ...

2. Students with a not-yet-complete understanding of high school algebra commonly argue that $(x+y)^2 = x^2 + y^2$. Below are three different approaches a teacher can use to help students see that this is not true.

   - One approach is to choose particular values for a and b, say $a = 1$ and $b = 2$, and ask students to plug those values into the left-hand side of the equation and simplify and then plug those same values into the right-hand side of the equation and simplify. Students can compare their answers to see that they are not equal. This approach can quickly show students that $(x+y)^2 \neq x^2 + y^2$ but doesn't give them insight into why the equality does not hold.
   - Another approach is to create an area model, such as the one below, which shows students that $(x+y)^2 \neq x^2 + y^2$. In this case the binomial $x+y$ is represented as a segment of length $x+y$, and the product $(x+y)^2$ is represented as the area of a rectangle whose side lengths are both $x+y$. This representation shows students which mathematical components they are missing in their answer, specifically the term $2xy$.

	x	y
x	x^2	xy
y	yx	y^2

   - A third approach is to apply the binomial theorem to $(x+y)^2$.

   (a) Use the binomial theorem to explain to a high school student why $(x+y)^2 \neq x^2 + y^2$.

   (b) Compare the "area model" approach with the "binomial theorem" approach. For what kinds of problems would each approach work? What insight does each approach highlight that will help students understand why $(x+y)^2 \neq x^2 + y^2$?

3. Charlie incorrectly expanded $(x+y)^2$ as follows.

   $$\begin{aligned}(x+y)^2 &= 1x^2y^2 + 2x^1y^1 + 1x^0y^0 \\ &= x^2y^2 + 2xy + 1\end{aligned}$$

   (a) What does Charlie understand about the binomial theorem?

   (b) What does Charlie not yet understand about the binomial theorem?

   (c) Write at least two questions that will help Charlie revise their work and develop a deeper understanding of the binomial theorem. Explain how those questions will help Charlie.

NAME: **ASSESSMENT PROBLEMS:** BINOMIAL THEOREM (page 1 of 2)

1. Consider the expansion of $(x+y)^{10}$.

   (a) Beyond "because the binomial theorem says so," explain why the coefficient of the x^4y^6 term is $\binom{10}{6}$.

   (b) One student states that the coefficient of the x^4y^6 term is $\binom{10}{4}$, and a second student states it is $\binom{10}{6}$. Beyond "because of symmetry," explain why each of them correctly computes the coefficient of x^4y^6.

ASSESSMENT PROBLEMS: BINOMIAL THEOREM (page 2 of 2)

2. Tencha, a high school student, expanded $(x-3y)^4$ using the binomial theorem and made some errors. Below is their work.

$$(x-3y)^4 = x^4 - 12x^3y - 18x^2y^2 - 12xy^3 - 3y^4$$

   (a) What does Tencha understand about the binomial theorem?

   (b) What does Tencha not yet understand about the binomial theorem?

   (c) Write two questions you can ask Tencha to help them revise their work. Explain how your questions could help guide their mathematical understanding.