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Foundations of Divisibility

Discrete Mathematics or Introduction to Proof

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1.1 Overview and Outline of Lesson

The divisibility of integers is a typical early focus in courses that introduce undergraduate students to mathematical proof, primarily because integers are familiar compared to more abstract constructs, but also because school students begin their formal study of divisibility of integers as early as elementary school. This lesson guides undergraduates to provide explanations about divisibility properties of integers using the definition of divisibility, the quotient-remainder theorem, and knowledge about the base-ten representation of integers. The lesson provides a rich opportunity for "looking back and looking forward," as undergraduates examine methods and reasoning about divisibility that are used in elementary grades and make sense of these methods using number theoretic techniques by applying theorems common in undergraduate study.

The focus of the Class Activity in this lesson is to identify the distinction between **what it means** for an integer m to be divisible by an integer n and **how one can determine** by examining an integer m if it is divisible by a certain integer n (see Beckmann, 2018, Chapter 8).

A note about the terminology we use in this lesson: School students often refer to "numbers" instead of the more precise "natural numbers" or "integers." This lesson uses hypothetical school students as characters in various problems, and we use colloquial language when conveying their ideas. We use formal mathematical terminology when posing problems for undergraduates.

1. Launch—Pre-Activity
   Prior to the lesson, undergraduates complete a Pre-Activity where they consider how a fifth-grade student might approach justifying why an integer is or is not divisible by 5 by using the structure of the base-ten system. Undergraduates use integer division with remainders and the representation of integers in base ten to establish the well-used test for divisibility by 5.

2. Explore—Class Activity

   - *Problem 1—Divisibility by 3:*
     The proof for determining if an integer is divisible by 3 has a different structure than the proof for determining

if an integer is divisible by 5. Undergraduates will develop a pictorial and numerical argument about why a particular three-digit integer is not divisible by 3 in order to establish a test for divisibility by 3.

- *Problem 2—Divisibility by 4:*
  Undergraduates consider the advantages of two different tests for divisibility by 4.
- *Problem 3—Student Reasoning About Multiples:*
  Undergraduates practice explaining to a hypothetical student why some multiples of 3 are even and others are odd.

3. Closure—Wrap-Up
   Conclude the lesson by discussing how “looking back” at school students’ reasoning about divisibility has enabled undergraduates to “look forward” to some tools of number theory and find important mathematical reasoning in school students’ ideas. If you want undergraduates to prove any of the tests for divisibility, these can be assigned as homework.

1.2 Alignment with College Curriculum

The divisibility of integers is often taught in a discrete mathematics or introduction to proof course. A study of tests for divisibility offers an opportunity for undergraduates to apply direct proof techniques that rely on properties of the integers. This lesson may be appropriate as part of a lesson on the quotient-remainder theorem and comes after undergraduates have learned and applied the definition of divisibility of integers.

1.3 Links to School Mathematics

This lesson addresses several mathematical knowledge and practice expectations included in school standards documents, such as the Common Core State Standards for Mathematics (CCSSM, 2010). The place value system is central in K–12 mathematics, and understanding divisibility of the integers is fundamental to understanding prime factorization, greatest common factors, and least common multiples. This lesson guides undergraduates to provide explanations about divisibility properties of integers using the definition of divisibility, the quotient-remainder theorem, and knowledge about the base-ten representation of integers.

This lesson highlights:

- Proofs of various tests for divisibility;
- “Looking for and making use of structure,” a K–12 mathematical practice (CCSSM, 2010), as undergraduates provide explanations for tests for divisibility that rely on the structure of the base-ten representation of integers.

1.4 Lesson Preparation

Prerequisite Knowledge

Undergraduates should know:

- Basic proof-writing techniques;
- The structure of an if-and-only-if proof;
- The definition of divisibility of integers;
- The quotient-remainder theorem.

Learning Objectives

In this lesson, undergraduates will encounter ideas about teaching mathematics, as described in Chapter 1 (see the five types of connections to teaching listed in Table 1.2). In particular, by the end of the lesson undergraduates will be able to:

- Reason about mathematical arguments for tests for divisibility;
- Prove some tests for divisibility and explain why they work;

- Attend to the base-ten representation of integers in their proofs;
- Apply the quotient-remainder theorem in their proofs;
- Analyze hypothetical school student work that investigates different tests for divisibility and evaluate questions one might ask hypothetical students to help guide their understanding about divisibility.

Anticipated Length

One 50-minute class session.

Materials

The following materials are required for this lesson.

- Pre-Activity (assign as homework prior to Class Activity)
- Class Activity (print Problems 1 and 2–3 to pass out separately)
- Homework Problems (assign at the end of the lesson)
- Assessment Problems (include on a quiz or exam after the lesson)

All handouts for this lesson appear at the end of this lesson, and LaTeX files can be downloaded from INSERT URL HERE.

1.5 Instructor Notes and Lesson Annotations

Before the Lesson

Assign the Pre-Activity as homework to complete in preparation for the lesson, and ask undergraduates to bring their solutions to class on the day you start the Class Activity. The problem on the Pre-Activity sets the stage for the rest of the lesson, with a focus on making sense of school student reasoning using the tools of discrete mathematics.

Pre-Activity Review (10 minutes)

Before reviewing the solutions to the Pre-Activity as a class, ask undergraduates when they have used tests of divisibility and then discuss the following connection to teaching.

Discuss This Connection to Teaching

- Middle school and high school students might use tests of divisibility when working on problems such as factoring, finding the greatest common divisor, and finding the least common multiple, for example when operating with fractions.
- Students in middle school and high school often memorize tests for divisibility but do not understand why they work. Prospective teachers should understand how these tests for divisibility are the result of structured mathematical reasoning.

As needed, engage in a whole class discussion to examine Rickie's argument and propose a generalization of it. This discussion can focus on the distinction between using the definition of divisibility to prove whether or not a particular integer is divisible by 5, which is what Rickie has developed for the integer 243, and generating a proof of the test for divisibility by 5, which examines the ones digit of any integer written in base ten.

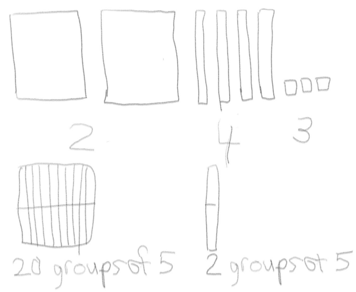
Pre-Activity

A fifth-grade class is discussing divisibility by 5. One student says, "My sister said you can tell if a number is divisible by 5 if it ends in a 0 or a 5." Another student adds, "Yeah, you can tell from skip counting by 5s: $5, 10, 15, 20, 25, \ldots$"

The teacher recognizes an opportunity to engage students in reasoning about properties of positive integers. The students have been using base-ten blocks to demonstrate whether various numbers are even or odd, so the teacher asks students to use their blocks to demonstrate whether the three numbers $243, 240$, and 245 are divisible by 5.

Notice that there is a difference between **how to recognize** if an integer is divisible by 5 or not, which involves inspecting the digit in the ones place, and **proving** that an integer is divisible by 5, which involves determining whether there exists an integer m such that the integer can be written as $5m$.

Below is what one student, Rickie, wrote to explain that 243 is not divisible by 5, but 240 and 245 are.



> *I represented 243 as 2 groups of 100, 4 groups of 10, and then 3 ones. Each group of 10 forms 2 groups of 5, and each group of 100 forms 20 groups of 5. Then you have to look at the leftover blocks.*
> *If the number ends in 0, then you haven't added any blocks and you can keep the groups you have. That's what happens with 240.*
> *If the number ends in 1, 2, 3, or 4, then you have leftover blocks. That's what happens with 243.*
> *If the number ends in 5, then you have 5 leftover blocks and you can put those into 1 group of 5. That's what happens with 245.*
> *I guess you could also have 6, 7, 8 or 9 leftover blocks, but that's it.*

State Rickie's test for divisibility by 5 as a biconditional statement. Explain how to generalize Rickie's argument about why the divisibility rule for 5 will always work.

Solution:
An integer is divisible by 5 if and only if the last digit of the integer is a 5 or a 0.
As a consequence of the quotient-remainder theorem, any integer n can be written as $n = 10q + r$, where q and r are integers and $0 \leq r \leq 9$. Because $10q = 5(2q)$ is always divisible by 5, divisibility of n by 5 is completely determined by r. If r is 0 or 5, r (and therefore n) is divisible by 5. Otherwise, r (and therefore n) is not divisible by 5.

Commentary:
Undergradates' explanations may be much less formal than what we indicate here, and you can use their responses to this introductory problem to gauge how to focus the class discussions in this lesson. Rickie's assertion that you can list all of the possibilities for the single blocks in a number represented by base-ten blocks is a consequence of the quotient-remainder theorem (or integer division with remainders) upon which a formal proof relies. If undergraduates don't name this, you can address it with a brief reminder.

Theorem 1 (Quotient-Remainder Theorem)**.** *Given any integer n and a positive integer d, there exist unique integers q and r such that $n = dq + r$ and $0 \leq r < d$.*

After discussing Rickie's argument, emphasize that the divisibility test for 5 consists of a biconditional statement, so there are two statements that undergraduates must justify: "If the last digit of an integer is 0 or 5, then the integer is divisible by 5." and "If an integer is divisible by 5, then the last digit of the integer is either 0 or 5." Undergraduates should recognize that a formal proof requires that they establish the certainty that "divisibility of n by 5 is completely determined by r." They will have the chance to examine this further in the Class Activity and Homework Problems.

Emphasize that in the proof for divisibility by 5, undergraduates make use of the base-ten representation of integers and the ideas about integer division with remainders that are encapsulated in the quotient-remainder theorem. Discuss the following connection to teaching.

Discuss This Connection to Teaching

Examining the work from this fifth-grader is a way to introduce the representation of integers in the base-ten system that is both useful to undergraduates in writing proofs but that is also accessible to younger students. Younger students will explain tests for divisibility by considering base-ten representations of integers and by considering what happens when tens, hundreds, thousands, and so on are divided by the integer in question (Beckmann, 2018). With base-ten blocks, students will commonly say or draw, for example, "2 groups of 100, 4 groups of 10, and 3 groups of 1" to represent 243. This emphasis on base-ten representation, both physically and numerically, prepares prospective teachers to be aware of what their students learn prior to entering the secondary grades.

Class Activity: Problem 1 (10 minutes)

As you pass out **Problem 1** of the Class Activity, let undergraduates know they will continue to examine hypothetical student work to explore various tests of divisibility, and then discuss the following connection to teaching.

Discuss This Connection to Teaching

- In order to develop the skill of understanding how others use and reason with mathematics, undergraduates will consider what aspects of school student work make it mathematically valid.
- Exploring why divisibility rules work before writing formal proofs will help undergraduates understand the underlying mathematical ideas before engaging in the details of rigorous proof. It will also help prospective teachers explain mathematical ideas that are accessible to their future students.

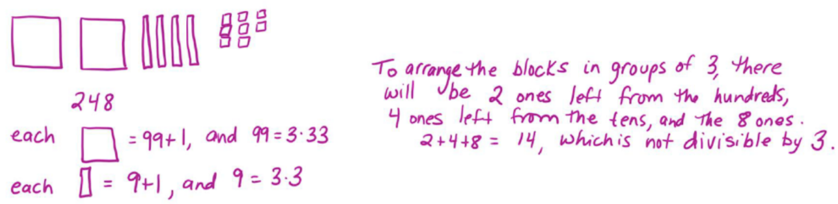
Instruct undergraduates to work in small groups on Problem 1 and then discuss the solutions. See Chapter 1 for guidance on facilitating group work and selecting and sequencing student work for use in whole-class discussion. From our experience, undergraduates may struggle with how to arrange 10 or 100 so that it is useful in recognizing groups of 3. Decide how long to let undergraduates wrestle with this idea as you circulate the class, helping them to realize that $10 = 9 + 1$, $100 = 99 + 1$, etc. It may be useful to have students focus on the tens digit in order to recognize groups of 3.

Class Activity Problem 1 : Part a

1. **Test for Divisibility by 3:**
   An integer is divisible by 3 if and only if the sum of its digits is divisible by 3.

   (a) Use a sketch to demonstrate how base-ten blocks could show whether or not the integer 248 is divisible by 3.

Sample Response:



For Problem 1(b), some undergraduates may not understand how to numerically express an argument, and it may be helpful to start this solution together (e.g., writing $248 = 2(100) + 4(10) + 8(1)$). As you circulate the class, help undergraduates to use their reasoning from part (a).

Class Activity Problem 1 : Part b

(b) Write a sequence of equivalent equations that numerically express the argument you made above with base-ten blocks.

Sample Response:

$$\begin{aligned} 248 &= 2(100) + 4(10) + 8(1) \\ &= 2(99+1) + 4(9+1) + 8 \\ &= (2 \times 99 + 2) + (4 \times 9 + 4) + 8 \\ &= (2 \times 99 + 4 \times 9) + (2 + 4 + 8) \\ &= (2 \times 3 \times 33 + 4 \times 3 \times 3) + (2 + 4 + 8) \end{aligned}$$

The first quantity, $(2 \times 3 \times 33 + 4 \times 3 \times 3)$, is a multiple of 3. The second quantity, $(2+4+8)$, is the sum of the digits in 248, but $(2+4+8) = 14$ is not a multiple of 3, so 248 is not divisible by 3.

After discussing the solution to Problem 1(b), tell undergraduates that they wrote 248 in expanded form, and emphasize the following connection to teaching.

Discuss This Connection to Teaching

The base-ten and place value systems are central in K–12 mathematics. Writing integers in expanded form is a part of the elementary school mathematics curriculum, so all prospective teachers should be familiar with writing integers in expanded form.

In generalizing this argument of divisibility by 3 in Problem 1(c), undergraduates may need a hint from you in order to recognize that they can use variables for digits and write a generic three-digit integer in base-ten representation as $abc = a(100) + b(10) + c(1)$.

Class Activity Problem 1 : Part c

(c) Outline an algebraic argument that shows that the test for divisibility by 3 holds for any three-digit integer. Ensure you consider both directions of the biconditional statement.

Solution:
Consider a three-digit integer n with digits a, b, and c. That is, a, b, and c are integers where $0 < a \leq 9$, $0 \leq b \leq 9$, and $0 \leq c \leq 9$. Thus,

$$\begin{aligned} n &= a(100) + b(10) + c(1) \\ &= a(99+1) + b(9+1) + c \\ &= (a \times 99 + b \times 9) + (a + b + c) \\ &= (a \times 3 \times 33 + b \times 3 \times 3) + (a + b + c) \\ &= 3t + (a + b + c) \end{aligned}$$

where $t = 33a + 3b$ is an integer because integers are closed under addition and multiplication.

If $a + b + c$ is divisible by 3, then the integer n is divisible by 3 (proving the direction, "If the sum of the digits is divisible by 3, then the integer is divisible by 3."). If the integer n is divisible by 3, then $3t + (a + b + c) = 3m$ for some integer m, and $a + b + c$ must also be divisible by 3 (proving the direction, "If the integer is divisible by 3, then the sum of the digits is divisible by 3.").

Commentary:
To generalize this argument for an integer with any number of digits, consider an integer with digits a_i in the 10^i place, and rewrite 10^i as $(10^i - 1) + 1$. You may want to give undergraduates this hint if you assign the homework problem where they prove that the test for divisibility by 3 holds for any integer.

Class Activity: Problems 2 & 3 (20 Minutes)

Distribute **Problems 2 and 3** of the Class Activity. Instruct undergraduates to work on the problems in small groups and discuss their solutions. These problems focus undergraduates' attention on responding to others' mathematical conjectures, which you can emphasize by discussing the following connection to teaching.

Discuss This Connection to Teaching

Problems 2 and 3 focus on analyzing other students' thinking in order to develop undergraduates' skills in understanding school student thinking and guiding school students' understanding. All undergraduates (especially prospective teachers) should explore how others use, reason with, and communicate mathematics. These problems also give prospective teachers (and tutors and future graduate students) an opportunity to think about how they would respond to student work in ways that nurture students' assets and understanding and in ways that help develop students' mathematical understanding.

Problem 2 distinguishes between how **one can recognize** whether an integer is divisible by 4 (Colby's method) and how to **prove** that an integer is divisible by 4 (Quinn's method).

Class Activity Problem 2

2. **Test for Divisibility by 4:**
An integer is divisible by 4 if and only if the integer formed by the last two digits (that is, the integer between 0 and 99) is divisible by 4.

Colby provided the following argument for explaining why the divisibility test for 4 always works.

> *The* 100 *block is always divisible by* 4*, because* $4 \times 25 = 100$*. Any number of* 100 *blocks will be divisible by* 4*. So I just have to check whether the number formed by the tens and ones is divisible by* 4*.*

Quinn prefers to use divisibility by 2 to check for divisibility by 4.

> *First I check if the number is even, and if it is, then I know it is divisible by* 2*. So I divide the number by* 2*, and if that answer is even, then I know the number is divisible by* 4*.*

In Problems 2(a) and 2(b), undergraduates compare the relative advantages of Colby's method and Quinn's method for determining which integers are divisible by 4.

Class Activity Problem 2 : Parts a & b

(a) Explain when Colby's method might have an advantage over Quinn's when trying to recognize if a four-digit integer is divisible by 4.

Sample Responses:

- Colby's rule might be faster than Quinn's if you can recognize which integers between 0 and 99 are divisible by 4.
- Colby's rule works well for students who know all of the two-digit multiples of 4.
- Quinn's method can take longer because you have to divide twice, although dividing by 2 isn't too hard.
- Colby's rule explains why it is true in a physical sense, because it relies on building the integer with base-ten blocks and then forming groups of 4. It seems like young students would understand it this way.
- Quinn's method could be more difficult because you have to think about all of the digits in an integer. With Colby's method, you only have to look at the last two digits of an integer.

(b) Explain why Quinn might have suggested this method of checking for divisibility by 4 while studying prime factorization.

Sample Responses:

- When finding prime factors, Quinn would recognize that if there are two 2's in the prime factorization of an integer, then 4 is a factor of the integer.
- Factorization and divisibility are connected ideas, because if 4 is a factor of an integer n, then $n = 4m$ for some integer m, which is equivalent to saying n is divisible by 4. Quinn might not say it that way though, depending on how old they are.
- Quinn hasn't really found a test for divisibility by 4, but Quinn is applying the definition of divisibility accurately.

In Problem 2(c), undergraduates have the opportunity to critically analyze the relative advantages of tests for divisibility. This can help them see that though the test for divisibility by 4 has a straightforward proof that relies on the definition of divisibility and integer division with remainders, it isn't as useful as the test for divisibility by 5 because multiples of 4 aren't as easy to recognize as multiples of 5.

Class Activity Problem 2 : Part c

(c) Explain why students might have an easier time determining whether an integer is divisible by 5 than determining whether an integer is divisible by 4.

Sample Responses:

- It's not as easy to recognize which integers between 0 and 99 are divisible by 4, but it is really easy to recognize which integers between 0 and 9 are divisible by 5.

- Students only need to look at the last digit and see if it's a 0 or 5 to determine whether the integer is divisible by 5. The divisibility rule for 4 is harder because you have to look at the last two digits of an integer and then think about whether it's divisible by 4.

Commentary:
The proof for a test of divisibility by 4 proceeds similarly to the proof for divisibility by 5, that is, writing an integer n as $n = 100q + r$, where r is an integer between 0 and 99. Though the proof is no more difficult than the proof for divisibility by 5, the resulting test is more difficult to apply. It can be useful for undergraduates to recognize the distinction.

Problem 3 prompts undergraduates to examine another hypothetical student's work. Parts (a) and (b) of this problem give them an opportunity to construct proofs using the definitions of even and odd.

Class Activity Problem 3 : Parts a & b

3. **Student Reasoning About Multiples:**
A student, Malik, tells his teacher that he has noticed that "when you skip count by 3s, you get the pattern odd-even-odd-even-odd-even, but when you skip count by 2s or 4s, you only get evens." Malik asks why that happens.

(a) Explain why skip counting by 2s (as in, "$2, 4, 6, 8, 10, \ldots$") or 4s (as in, "$4, 8, 12, 16, 20, \ldots$") yields only even integers. Use the definition of even integers in your explanation.
Sample Response:
Skip counting by 2s results in integers of the form $2i$, where i is an integer. These integers are even by the definition of even integer. Skip counting by 4s results in integers of the form $4j$, where j is an integer. $4j = 2(2j)$, which is even by the definition of even integer (since $2j$ must also be an integer).

(b) Explain why skip counting by 3s (as in, "$3, 6, 9, 12, 15, 18, \ldots$") yields both odd and even integers.
Sample Response:
Skip counting by 3s results in integers of the form $3i$, where i is an integer. If i is odd, then there exists an integer n such that $i = 2n + 1$. Then

$$3i = 3(2n + 1) = 6n + 3 = 2(3n) + 2 + 1 = 2(3n + 1) + 1,$$

which is odd. If i is even, then there exists an integer n such that $i = 2n$ and

$$3i = 3(2n) = 2(3n),$$

which is even.

Problems 3(c) and 3(d) introduce undergraduates to guiding the mathematical thinking of others by evaluating mathematical questions that can be posed to students. The prompts posed here encourage the use of drawings as part of building mathematical reasoning.

Class Activity Problem 3 : Parts c & d

(c) Explain how the two sketches below, representing 12 and 15, might help Malik to understand why the pattern holds.

Sample Response:
By arranging the blocks with the groups of 6 made obvious, a student could see that a number that can be arranged by both groups of 3 and groups of 2 will be even. This is the case for the number 12. If the arrangements of blocks has a group of three "left over" (like 15) then the number is a multiple of 3 and is odd.

(d) Explain how the following question might help Malik to understand why the pattern holds.

If I skip count by 3*s and the last number I said was an even number, will the next multiple of* 3 *be even or odd? Draw a picture to show why.*

Sample Responses:

- By having a student draw a picture, the student will see that if they have an integer already grouped by 2s, adding another group of 3 will show that the integer is odd.
- By having the student focus on what happens when skipping from a multiple of three that is even to the next multiple of three, it simplifies the reasoning by focusing on one of the two possible cases.

After discussing the solutions to Problems 3(c) and 3(d), tell undergraduates that Malik's reasoning is an example of how looking for structure one way (i.e., the numerical structure that results when dividing by 2) can lead to finding structure in other ways—in Malik's case, he doesn't have a name for it, but he's recognized that multiples of 3 can be classified by whether they are even or odd. This leads directly to a notion of divisibility by 6, and undergraduates will further consider divisibility by 6 in the Homework Problems.

Emphasize the following connection to teaching.

Discuss This Connection to Teaching

School students often make mathematical observations. It is valuable for teachers to emphasize that mathematical reasoning and proof are what is used to establish the truth of a conjecture, and it is in the student's power to establish that a statement is either true or false. By using questioning to guide Malik's understanding, teachers are developing his mathematical reasoning abilities. Teachers help students by recognizing what the students understand and identifying the important mathematical ideas the students are trying to communicate. Students learn that the mathematics, rather than the teacher, is the authority of mathematical truth.

Wrap-Up (up to 10 minutes)

Conclude the lesson by briefly discussing the similarities and differences in the proofs of the divisibility tests for 5, 3, and 4. This discussion may include the following ideas:

- Divisibility tests for 5 and 4 are similar, because 5 is a factor of any number of 10s and 4 is a factor of any number of 100s. Thus, these tests can proceed by inspecting the digits in certain place values.

- Three is not a factor of any power of 10, so the structure of the divisibility test for 3 is different than for 5 and 4. Writing the integer in expanded form is useful, because each power of 10 can be written as $(10^k - 1) + 1$, and $10^k - 1$ is divisible by 3.

Summarize a few connections to teaching that were present throughout the lesson, focusing on how undergraduates engaged in problems that focused on analyzing student thinking and guiding student understanding.

Discuss This Connection to Teaching

- Undergraduates made use of the structure of integers in the base-ten system, relying on place value representation and closure of integers under various addition and multiplication.
- Looking for and making use of structure is a mathematical practice exercised by students and by mathematicians.
- The problems in this lesson demonstrate how formal proofs underlie the mathematical reasoning of younger students.

We have found it useful to ask undergraduates to complete an exit ticket at the end of the lesson. (See Chapter 1 for guidance on using exit tickets.)

Homework Problems

At the end of the lesson, assign the following homework problems, and include any additional homework problems at your discretion. The first two problems highlight connections to teaching and the third problem gives undergraduates an opportunity to write formal proofs of the divisibility rules they encountered throughout the lesson.

Problem 1 builds from the "Malik" problem from the Class Activity and prompts undergraduates to consider the validity of two hypothetical students' tests for divisibility by 6. High school students may find or develop divisibility rules and prospective teachers will need to be able to consider ideas that students have and assess whether their ideas are sometimes or always true. This problem relies on the mathematical practice of constructing viable arguments and critiquing the reasoning of others.

Homework Problem 1

1. Adam and Charlotte learn that a number is divisible by 6 if it is divisible by both 2 and 3. Each attempts to apply similar reasoning to state a divisibility rule for 20. Adam says that "because $20 = 2 \times 10$, if a number is divisible by both 2 and 10, then the number is divisible by 20." Charlotte states that "because 20 is divisible by 4 and 5, if a number is divisible by both 4 and 5, then the number is divisible by 20."

   (a) Why doesn't Adam's rule work, but Charlotte's rule does? What is the key difference between their two rules?

   Sample Responses:

   - Adam's rule does not work because if a number is divisible by 10, then you know it is divisible by 2; checking to see if it is divisible by 2 after knowing it is divisible by 10 does not give us any new information.
   - Charlotte's rule works because 4 and 5 are relatively prime.
   - 4 and 5 share no common factors but 2 and 10 share a factor of 2. As long as the factors are relatively prime, this method will work.
   - Adam might be confusing the statement with its converse. It is true that if a number is divisible by 20, then it is divisible by 2 and 10, but that's the converse of the rule he stated.

(b) Adam's proof to their conjecture is shown below. Identify the error in their proof and explain why it is an error.

> Let n be any integer that is divisible by 2 and 10. By the definition of divisibility, since n is divisible by 2, there is an integer k where $n = 2k$. Since n is also divisible by 10, that means that k must be divisible by 10. By the definition of divisibility, there is an integer l where $k = 10l$. Using substitution, $n = 2k = 2(10l) = 20l$. Since $n = 20l$ for some integer l, n is divisible by 20.

Solution:
The error occurs in the following sentence: "Since n is also divisible by 10, that means that k must be divisible by 10." Because n is divisible by 10, there must be an integer m such that $n = 10m$. Adam could conclude that $2k = 10m$ so k is divisible by 5, but not by 10.

(c) Their teacher recognizes that Adam was probably testing the number 20 or 40 when they conjectured their rule. What are some other integers the teacher could encourage Adam and Charlotte to experiment with that would help them to understand divisibility by 20?

Sample Response:
Any number that is a multiple of 10 but not a multiple of 4 can show what Adam's mistake is. Numbers like 30 and 50 fit the hypothesis of Adam's rule but not the conclusion and are good numbers for them to investigate.

Problem 2 highlights another hypothetical student's conjecture about divisibility tests, this time about a divisibility test for 7.

Homework Problem 2

2. A student, Isla, tells you that she has created a "test" for divisibility by 7. She claims that an integer n is divisible by 7 if and only if the rightmost two digits of n form an integer that is a multiple of 7. Provide a counterexample showing that Isla's test for divisibility by 7 doesn't work, and explain why Isla may believe her test works.

Sample Responses:

- 114 has rightmost two digits that are divisible by 7, but 114 is not divisible by 7.
- 105 is a multiple of 7 (i.e., $7 \times 15 = 105$) but its rightmost two digits form an integer (5) that is not divisible by 7.
- Isla might believe this rule is true because she is copying the format of the test for divisibility by 4. She still needs to learn why the rule is true for testing divisibility by 4.

Problem 3 prompts undergraduates to write formal proofs of the divisibility rules for 5, 3, 4, and 6. Proofs for these statements are generalizations of the reasoning presented in class. Writing formal proofs of these statements can give undergraduates practice with proving biconditional statements, and they may need prompting to write these tests as biconditional statements.You might choose to assign only some of these four, depending on your course goals. The style and rigor of these proofs should follow your expectations for other proofs in your course.

Homework Problem 3

3. Proving Tests for Divisibility

(a) Prove that an integer is divisible by 5 if and only if its last digit is 0 or 5.

Commentary:
This proof was outlined during the Class Activity. We want undergraduates to recognize that any integer n can be written $n = 10q + r$, where q is an integer and r is an integer such that $0 \leq r \leq 9$, and that the base-ten representation of integers is such that r is the ones digit of the integer n.

(b) Prove that an integer is divisible by 3 if and only if the sum of its digits is divisible by 3.

Commentary:
The proof of this statement for a 3-digit integer was outlined during the Class Activity. To generalize an argument for an integer with any number of digits, consider an integer with digits a_i in the 10^i place.

(c) Prove that an integer is divisible by 4 if and only if the integer formed by its last two digits is divisible by 4.

Commentary:
This proof was outlined during the Class Activity. We want undergraduates to recognize that any integer n can be written $n = 100q + r$, where q is an integer and r is an integer such that $0 \leq r \leq 99$, and that the base-ten representation of integers is such that r is the integer formed by the tens and ones digits of the integer n. The structure of this argument otherwise mimics that of part (a).

(d) Prove that an integer is divisible by 6 if and only if the integer is divisible by both 2 and 3.

Commentary:
This problem provides undergraduates an opportunity to prove that even multiples of three, which they examined using Malik's skip counting observation, are divisible by 6. The proof of the statement "If an integer is divisible by 6, then it is divisible by both 2 and 3." is straightforward. To prove the other direction, there are a few approaches that naturally arise after undergraduates have engaged in the reasoning from this lesson. One approach recognizes that using 6 as the divisor in the quotient-remainder theorem is useful: From the quotient remainder theorem, $n = 6q + r$ for some integer q and some integer r such that $0 \leq r \leq 5$. Also, if an integer is divisible by both 2 and 3, then it satisfies $n = 2k$ and $n = 3l$ for some integers k and l. Since $n = 2k$, it follows that r is a multiple of 2. Since $n = 3l$, it follows that r is a multiple of 3. The only value of r in $\{0, 1, 2, 3, 4, 5\}$ that is both a multiple of 2 and a multiple of 3 is 0, so $n = 6q$ and n is divisible by 6. A second approach relies on the definition of divisibility and the closure of the integers: $n = 2k$ and $n = 3l$ for some integers k and l. It follows that $3n = 6k$ and $2n = 6l$, so subtracting these equations from the left and right sides, respectively, yields $n = 6(k - l)$, where $k - l$ is an integer and thus, n is divisible by 6.

Assessment Problems

The following two problems address ideas explored in the lesson, with a focus on connections to teaching and mathematical content. You can include these problems as part of your usual course quizzes or exams.

Problem 1 assesses undergraduates' proof-writing techniques and how they relied on properties of integers in their proof.

Assessment Problem 1

1. An integer n is divisible by 10 if and only if the final digit of the integer is 0.

   (a) Prove that this is true.

   Sample Response:

   Since this is a biconditional statement, the proof involves two statements.

   Claim 1. *If n is divisible by 10, then the final digit of n is 0.*

   Proof. Let n be an integer that is divisible by 10. This means that $n = 10k$ for some integer k. By the quotient-remainder theorem, we also know that $n = 10q + r$ for some integer q and some integer r, where $0 \leq r \leq 9$. By the base-ten representation of integers, r is the final digit of n. Thus,

$$10k = 10q + r$$
$$10k - 10q = r$$
$$10(k - q) = r$$

   Because $k - q$ is an integer (since integers are closed under addition), r is a multiple of 10. The only multiple of 10 such that $0 \leq r \leq 9$ is 0, so r must be 0. □

   Claim 2. *If the final digit of n is 0, then n is divisible by 10.*

   Proof. Let n be an integer whose final digit is 0. By the quotient-remainder theorem, $n = 10q + r$ for some integer q and some integer r, where $0 \leq r \leq 9$. By the base-ten representation of integers, r is the final digit of n, so $r = 0$. Thus $n = 10q$, and n is divisible by 10. □

   (b) Describe how you relied on properties of integers in your proof.

   Sample Responses:

   - Combining integers via multiplication, subtraction, and addition always leads to other integers.
   - In the base-ten system of integers, the remainder from dividing n by 10 is the last digit of the numeral for n.

Problem 2 assesses undergraduates' ability to analyze school student thinking and to guide school student understanding. It also focuses their attention on the biconditional statement that underlies each test for divisibility.

Assessment Problem 2

2. In class, Olivia learned that a number is divisible by 6 if it is divisible by both 2 and 3 and used that to conjecture a divisibility rule for 60. Olivia says that because 60 is divisible by 6 and 10, you can tell which numbers are divisible by 60 by checking if the number is divisible by both 6 and 10.

   (a) Rewrite Olivia's conjecture as a biconditional statement.

   Sample Responses:

   - An integer is divisible by 60 if and only if it is divisible by both 6 and 10.
   - "If an integer is divisible by 60, then it is divisible by both 6 and 10." and "If an integer is divisible by both 6 and 10, then it is divisible by 60.".

(b) One direction of the biconditional statement is true and the other is false. State which direction is false and find a counterexample showing that it is false.

Solution:
This statement is false: If an integer is divisible by both 6 and 10, then it is divisible by 60. Many counterexamples exist. 30 is the smallest counterexample.

(c) Explain how the following question might help Olivia to understand that her rule doesn't always work.

What is the least common multiple of 6 *and* 10*?*

Sample Response:
It would help Olivia see that 30 is a number that is divisible by both 6 and 10 but isn't divisible by 60.

(d) Provide two reasons why you think Olivia made this conjecture.

Sample Responses:

- This sort of statement is true with other numbers. For example, it is true that if a number is divisible by both 2 and 5, then the number is divisible by 10. The distinction is that 2 and 5 don't share any common prime factors, but 6 and 10 share 2 as a common prime factor
- Olivia is only thinking about one direction of this biconditional statement. It is true that if a number is divisible by 60, then the number is divisible by 6 and 10.

1.6 References

[1] Beckmann, S. (2018). *Mathematics for elementary teachers with activities*. Pearson.

[2] National Governors Association Center for Best Practices & Council of Chief State School Officers (2010). *Common Core State Standards for Mathematics.* Authors. Retrieved from `http://www.corestandards.org/`

1.7 Lesson Handouts

Handouts for use during instruction are included on the pages that follow. LaTeX files for these handouts can be downloaded from INSERT URL HERE.

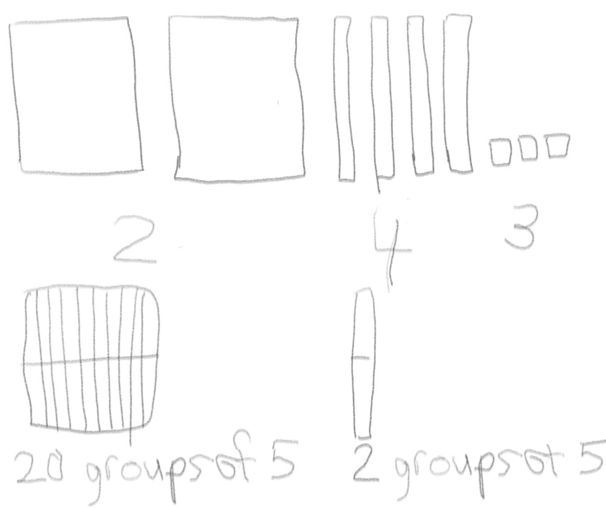
NAME: **PRE-ACTIVITY:** FOUNDATIONS OF DIVISIBILITY (page 1 of 1)

A fifth-grade class is discussing divisibility by 5. One student says, "My sister said you can tell if a number is divisible by 5 if it ends in a 0 or a 5." Another student adds, "Yeah, you can tell from skip counting by 5s: $5, 10, 15, 20, 25, \ldots$"

The teacher recognizes an opportunity to engage students in reasoning about properties of positive integers. The students have been using base-ten blocks to demonstrate whether various numbers are even or odd, so the teacher asks students to use their blocks to demonstrate whether the three numbers 243, 240, and 245 are divisible by 5.

Notice that there is a difference between **how to recognize** if an integer is divisible by 5 or not, which involves inspecting the digit in the ones place, and **proving** that an integer is divisible by 5, which involves determining whether there exists an integer m such that the integer can be written as $5m$.

Below is what one student, Rickie, wrote to explain that 243 is not divisible by 5, but 240 and 245 are.



I represented 243 as 2 groups of 100, 4 groups of 10, and then 3 ones. Each group of 10 forms 2 groups of 5, and each group of 100 forms 20 groups of 5. Then you have to look at the leftover blocks.
If the number ends in 0, then you haven't added any blocks and you can keep the groups you have. That's what happens with 240.
If the number ends in 1, 2, 3, or 4, then you have leftover blocks. That's what happens with 243.
If the number ends in 5, then you have 5 leftover blocks and you can put those into 1 group of 5. That's what happens with 245.
I guess you could also have 6, 7, 8 or 9 leftover blocks, but that's it.

State Rickie's test for divisibility by 5 as a biconditional statement. Explain how to generalize Rickie's argument about why the divisibility rule for 5 will always work.

NAME: **CLASS ACTIVITY:** FOUNDATIONS OF DIVISIBILITY (page 1 of 3)

1. **Test for Divisibility by 3:**
An integer is divisible by 3 if and only if the sum of its digits is divisible by 3.

(a) Use a sketch to demonstrate how base-ten blocks could show whether or not the integer 248 is divisible by 3.

(b) Write a sequence of equivalent equations that numerically express the argument you made above with base-ten blocks.

(c) Outline an algebraic argument that shows that the test for divisibility by 3 holds for any three-digit integer. Ensure you consider both directions of the biconditional statement.

2. **Test for Divisibility by 4:**
An integer is divisible by 4 if and only if the integer formed by the last two digits (that is, the integer between 0 and 99) is divisible by 4.

Colby provided the following argument for explaining why the divisibility test for 4 always works.

The 100 *block is always divisible by* 4*, because* $4 \times 25 = 100$*. Any number of* 100 *blocks will be divisible by* 4*. So I just have to check whether the number formed by the tens and ones is divisible by* 4*.*

Quinn prefers to use divisibility by 2 to check for divisibility by 4.

First I check if the number is even, and if it is, then I know it is divisible by 2*. So I divide the number by* 2*, and if that answer is even, then I know the number is divisible by* 4*.*

(a) Explain when Colby's method might have an advantage over Quinn's when trying to recognize if a four-digit integer is divisible by 4.

(b) Explain why Quinn might have suggested this method of checking for divisibility by 4 while studying prime factorization.

(c) Explain why students might have an easier time determining whether an integer is divisible by 5 than determining whether an integer is divisible by 4.

3. **Student Reasoning About Multiples:**
A student, Malik, tells his teacher that he has noticed that "when you skip count by 3s, you get the pattern odd-even-odd-even-odd-even, but when you skip count by 2s or 4s, you only get evens." Malik asks why that happens.

   (a) Explain why skip counting by 2s (as in, "$2, 4, 6, 8, 10, \ldots$") or 4s (as in, "$4, 8, 12, 16, 20, \ldots$") yields only even integers. Use the definition of even integers in your explanation.

   (b) Explain why skip counting by 3s (as in, "$3, 6, 9, 12, 15, 18, \ldots$") yields both odd and even integers.

   (c) Explain how the two sketches below, representing 12 and 15, might help Malik to understand why the pattern holds.

   (d) Explain how the following question might help Malik to understand why the pattern holds.
If I skip count by 3s and the last number I said was an even number, will the next multiple of 3 be even or odd? Draw a picture to show why.

NAME: **HOMEWORK PROBLEMS:** FOUNDATIONS OF DIVISIBILITY (page 1 of 1)

1. Adam and Charlotte learn that a number is divisible by 6 if it is divisible by both 2 and 3. Each attempts to apply similar reasoning to state a divisibility rule for 20. Adam says that "because $20 = 2 \times 10$, if a number is divisible by both 2 and 10, then the number is divisible by 20." Charlotte states that "because 20 is divisible by 4 and 5, if a number is divisible by both 4 and 5, then the number is divisible by 20."

   (a) Why doesn't Adam's rule work, but Charlotte's rule does? What is the key difference between their two rules?

   (b) Adam's proof to their conjecture is shown below. Identify the error in their proof and explain why it is an error.

   > Let n be any integer that is divisible by 2 and 10. By the definition of divisibility, since n is divisible by 2, there is an integer k where $n = 2k$. Since n is also divisible by 10, that means that k must be divisible by 10. By the definition of divisibility, there is an integer l where $k = 10l$. Using substitution, $n = 2k = 2(10l) = 20l$. Since $n = 20l$ for some integer l, n is divisible by 20.

   (c) Their teacher recognizes that Adam was probably testing the number 20 or 40 when they conjectured their rule. What are some other integers the teacher could encourage Adam and Charlotte to experiment with that would help them to understand divisibility by 20?

2. A student, Isla, tells you that she has created a "test" for divisibility by 7. She claims that an integer n is divisible by 7 if and only if the rightmost two digits of n form an integer that is a multiple of 7. Provide a counterexample showing that Isla's test for divisibility by 7 doesn't work, and explain why Isla may believe her test works.

3. Proving Tests for Divisibility

   (a) Prove that an integer is divisible by 5 if and only if its last digit is 0 or 5.

   (b) Prove that an integer is divisible by 3 if and only if the sum of its digits is divisible by 3.

   (c) Prove that an integer is divisible by 4 if and only if the integer formed by its last two digits is divisible by 4.

   (d) Prove that an integer is divisible by 6 if and only if the integer is divisible by both 2 and 3.

NAME: **ASSESSMENT PROBLEMS:** FOUNDATIONS OF DIVISIBILITY (page 1 of 2)

1. An integer n is divisible by 10 if and only if the final digit of the integer is 0.

   (a) Prove that this is true.

   (b) Describe how you relied on properties of integers in your proof.

2. In class, Olivia learned that a number is divisible by 6 if it is divisible by both 2 and 3 and used that to conjecture a divisibility rule for 60. Olivia says that because 60 is divisible by 6 and 10, you can tell which numbers are divisible by 60 by checking if the number is divisible by both 6 and 10.

   (a) Rewrite Olivia's conjecture as a biconditional statement.

   (b) One direction of the biconditional statement is true and the other is false. State which direction is false and find a counterexample showing that it is false.

   (c) Explain how the following question might help Olivia to understand that her rule doesn't always work.

   What is the least common multiple of 6 *and* 10*?*

   (d) Provide two reasons why you think Olivia made this conjecture.