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Groups of Transformations

Abstract (Modern) Algebra I

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1.1 Overview and Outline of Lesson

This lesson follows a typical introduction to the group axioms. Undergraduates investigate sets of matrices that represent familiar geometric transformations: rotations about the origin and reflections about lines through the origin. Working in $\mathbb{R}^2$, these sets are examined for group structure and commutativity under matrix multiplication by building on undergraduates' existing knowledge of the geometry of these types of transformations. This supports prospective teachers' mathematical knowledge for considering ways to justify (or redirect) certain intuitions high school geometry students may have about these transformations, in particular whether "order matters" when applying a sequence of transformations to an object.

1. Launch—Pre-Activity
   Undergraduates complete this assignment prior to the lesson. In it, undergraduates engage in ascribing mathematical formality to the idea of when "order matters" by considering the closure, associativity, and commutativity of the composition of rotations about the origin. Then, they explore two sets of 2×2 matrices by varying a parameter and observing how the corresponding matrices differently affect vectors in $\mathbb{R}^2$. These sets of matrices are characterized as rotations and reflections, setting the stage for the Class Activity.

2. Explore—Class Activity
   - *Problems 1 & 2*:
     The group axioms (and commutativity) are explored with respect to the set of rotations (about the origin) by making geometric interpretations of matrix equations. Undergraduates conclude that this set is an abelian group under matrix multiplication.
   - *Problems 3 & 4*:
     The group axioms (and commutativity) are explored with respect to the set of reflections (about lines through the origin) by reasoning geometrically about the existence of inverses and an identity element. Undergraduates conclude that this set is neither a group nor commutative under matrix multiplication.
   - *Problems 5 & 6*:
     The commutativity of elements in the set of rotations and reflections is explored by helping a hypothetical geometry student reason about the congruency of figures. Undergraduates conclude that the set of rotations and reflections are not commutative under matrix multiplication.

3. Closure—Wrap-Up
   The instructor wraps up the lesson by reviewing the two sets of matrices, which rigid motions they represent, and whether these sets are (abelian) groups under matrix multiplication. This information supports prospective teachers in discussions with their future students while considering whether "order matters" when applying rigid motions to geometric shapes.

1.2 Alignment with College Curriculum

Undergraduates explore sets of matrices that represent familiar geometric transformations for group structure, thereby adding an accessible, easily-visualized example of a (non-) group to their understanding of the topic. They are then asked to consider whether "order matters" when applying each type of transformation, laying the groundwork for discussions about the associativity and commutativity of binary operations.

1.3 Links to School Mathematics

Matrices are often presented in high school mathematics classrooms as arbitrary objects that calculators use to solve unwieldy systems of equations; on the other hand, rigid motions are used to establish the congruence of shapes but are not treated as especially formal mathematical objects. These shortcomings are addressed simultaneously by demonstrating that two rigid motions (reflections and rotations) can be represented as linear transformations on $\mathbb{R}^2$ via matrices. The sets of these matrix transformations, once explored for group structure, are used to interpret a high school geometry problem through the lens of abstract algebra.

This lesson highlights:

- Using group structure to formalize ideas from school mathematics, such as "undoing" an operation and establishing congruency.
- Connecting matrix transformations to familiar concepts from high school geometry.

This lesson addresses several mathematical knowledge and practice expectations included in high school standards documents, such as the Common Core State Standards for Mathematics (CCSSM, 2010). For example, it is expected that high school students learn to use matrix arithmetic and the underlying properties of those arithmetic operations. This includes the fact that matrix multiplication is not commutative, but is associative; that the identity matrix and the zero matrix are the matrix analogues of 1 and 0 in the real numbers, respectively; and that matrices can act as transformations on vectors under matrix multiplication (see CCSS.MATH.CONTENT.HSM.VM.C for a complete list of properties). High school students also work with rigid motions (although not usually in the form of matrices) in order to establish the congruency of triangles on a plane (c.f. CCSS.MATH.CONTENT.HSG.CO.B.7). Finally, this lesson emphasizes the need for viable mathematical arguments, encourages undergraduates to look for and make use of structural similarities, and provides opportunities to both critique the reasoning of others and to practice the appropriate transference of reasoning from one setting to another.

1.4 Lesson Preparation

Prerequisite Knowledge

Undergraduates should know:

- Matrix multiplication, including how a 2×2 matrix acts on a vector from $\mathbb{R}^2$;
- The definitions of a group and an abelian group.

Learning Objectives

In this lesson, undergraduates will encounter ideas about teaching mathematics, as described in Chapter 1 (see the five types of connections to teaching listed in Table 1.2). In particular, by the end of the lesson undergraduates will be able to:

- Decide whether sets of matrices are (abelian) groups under a particular binary operation;

- Explain whether "order matters" when considering a binary operation on a set;
- Translate the above conclusions into geometric language by describing the effect of (products of) matrices on a vector;
- Analyze hypothetical student work to evaluate reasoning about transformations.
- Pose guiding questions to help a hypothetical student connect geometric understandings about transformations to their assertions about whether corresponding sets of matrices are groups under a given binary operation.

Anticipated Length

One 75-minute class session.

Materials

The following materials are required for this lesson:

- Pre-Activity (assign as homework prior to Class Activity)
- Class Activity
- Homework Problems (assign at the end of the lesson)
- Assessment Problems (include on quiz or exam after the lesson)

All handouts for this lesson appear at the end of this lesson, and LaTeX files can be downloaded from INSERT URL HERE.

1.5 Instructor Notes and Lesson Annotations

Before the Lesson

Assign the Pre-Activity as homework to be completed in preparation for this lesson.

> *We recommend that you collect this Pre-Activity the day before the lesson so that you can review undergraduates' responses before you begin the Class Activity. This will help you determine if you need to spend additional time reviewing the solutions to the Pre-Activity with your undergraduates.*

Pre-Activity Review (10 minutes)

Briefly discuss your undergraduates' responses to each part of Problem 1. If appropriate for your class, you may wish to ask for some other examples of operations for which order does or does not matter. Make sure that undergraduates are specific about the set of objects on which the operation is acting. For example, "order doesn't matter" when multiplying integers, but it does when multiplying matrices.

> *If your class has already defined "order of a group" or "order of an element," you may wish to clarify here that the word "order" in the phrase "order matters" does not refer to either of these precise, mathematical definitions. Also, it is not exactly related to the "order of operations" either, since we are only considering one operation at a time. It is precisely this ambiguity that we hope to alleviate by defining, rigorously and mathematically, what we mean when we say "order matters."*
>
> *Undergraduates may be unclear exactly what is meant by "composition" in Problem 1. You may need to clarify that when we ask if rotations are closed under composition we really mean: can the combined act of rotating a vector twice about the origin be represented, in total, as one rotation?*

Pre-Activity Problem 1

Consider the sum $2 + 6 + 4 + 7$. When people say that "order does not matter" when computing such a sum they actually mean two things: the order of the individual terms of the sum can be rearranged without affecting the final result (for instance, $7 + 4 + 6 + 2$ and the original sum are sure to each give the same answer, 19) and, moreover, the order in which one chooses to compute the individual addition operations is unimportant (for instance, $((2+6)+4)+7$ and $2+((6+4)+7)$ both yield the same final result of 19). This conclusion relies on the three fundamental beliefs of integer arithmetic:

- Integer addition is **closed**; that is, $a+b$ is itself an integer for all integers a and b.
- Integer addition is **commutative**; that is, $a+b=b+a$ for all integers a and b.
- Integer addition is **associative**; that is, $(a+b)+c=a+(b+c)$ for all integers a, b, and c.

1. Consider the set of all rotations about the origin of the plane.
[Recall that transformations (e.g., rotations) are functions. As such, for rotations r_α and r_β on $\mathbb{R}^2$, the *composition* of r_α followed by r_β, $r_\beta \circ r_\alpha$ is defined by $r_\beta \circ r_\alpha(P) = r_\beta(r_\alpha(P))$ where $P \in \mathbb{R}^2$.]

(a) Is this set closed under composition? Explain.

Sample Response:
Yes. Rotating about the origin by some number of radians, followed by then rotating again by a different amount, is equivalent to rotating by the sum of the two angles of rotation. So the composition of two rotations is again a rotation.

(b) Do rotations commute with each other under composition? Explain.

Sample Response:
Yes. I can rotate a vector by two different amounts in either order and the resultant vector will be the same amount either way.

(c) Do rotations about the origin satisfy the associative law under composition? Explain.

Sample Responses:

- When rotating a vector by three different angles, the way that I group the rotations before doing them does not affect the overall rotation.
- Yes. If you associate a rotation with the corresponding angle measure of that rotation (for example, you treat a rotation of $\frac{\pi}{6}$ as just $\frac{\pi}{6}$), then the composition of rotations about the origin is the same as real number addition. Real number addition is clearly associative, so composition of angles must also be.

(d) Does "order matter" when performing a series of rotations about the origin in the plane? Explain.

Sample Response:
No. Rotations appear to be closed under composition and both commutative and associative with respect to composition.

Undergraduates may struggle to justify Problem 1(c), even imprecisely. One approach is to appeal to the natural isomorphism between $\mathbb{R}$ mod 2π under addition and the set of rotations about the origin under composition. Use a degree of formality when handling this relationship that is appropriate for your class—for example, you might choose to wait until the introduction of the set Σ in the next problem to make this isomorphism more explicit.

Pre-Activity Problem 2

2. Consider the set Σ, given below.

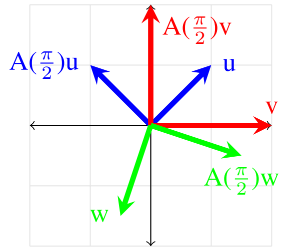
$$\Sigma = \left\{ \mathrm{A}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} : \theta \in \mathbb{R} \right\}$$

(a) Calculate $A(\frac{\pi}{2})$. Choose three nonzero vectors v_1, v_2, and v_3 in $\mathbb{R}^2$ that are not all scalar multiples of one another. Compute $A(\frac{\pi}{2})v_1$, $A(\frac{\pi}{2})v_2$, and $A(\frac{\pi}{2})v_3$. Sketch all six vectors on the same coordinate plane.

Sample Response:

First, note that $A(\frac{\pi}{2}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Then, when using $u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $v = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, and $w = \begin{bmatrix} -1/2 \\ -3/2 \end{bmatrix}$, we have:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$
$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1/2 \\ -3/2 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}$$

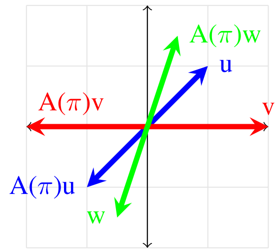


(b) Repeat the process of Problem 2(a) with $A(\theta)$ for a different nonzero value of θ and the same vectors.

Sample Response:

Choosing $A(\pi) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, with the same vectors as above:

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$
$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1/2 \\ -3/2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 3/2 \end{bmatrix}$$



(c) Write a geometric description of how an arbitrary matrix from Σ acts on vectors in $\mathbb{R}^2$ based on your sketches in Problems 2(a) and 2(b).

Solution:

Matrices of this type represent rotations about the origin. If θ is positive, then this appears as a counterclockwise rotation of θ radians; if θ is negative, then this appears as a clockwise rotation of $|\theta|$ radians.

If undergraduates are unsure about whether the vector's magnitude is preserved under these operations, encourage them to consider Problem 3 geometrically rather than through computations.

Pre-Activity Problem 3

3. Consider the set Φ, given below.

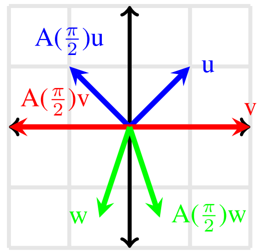
$$\Phi = \left\{ \mathrm{B}(\theta) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} : \theta \in \mathbb{R} \right\}$$

(a) Repeat the process of Problem 2(a) with matrix $B(\frac{\pi}{2})$.

Sample Response:

This time, note that $B(\frac{\pi}{2}) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Then, with the same vectors as above:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1/2 \\ -3/2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -3/2 \end{bmatrix}$$

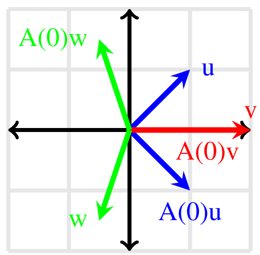


(b) Repeat the process of Problem 3(a) with $B(\theta)$ for a different value of θ and the same vectors.

Sample Response:

First, we choose $B(0) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Then, with the same vectors as above:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1/2 \\ -3/2 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 3/2 \end{bmatrix}$$



(c) Write a geometric description of how an arbitrary matrix from Φ acts on vectors in $\mathbb{R}^2$ based on your sketches in Problems 3(a) and 3(b).

Solution:

Matrices of this type represent reflections about lines through the origin. The line of reflection forms an angle of θ radians with the positive x-axis.

Give undergraduates a few minutes to compare their answers for Problems 2(c) and 3(c) in small groups. See Chapter 1 for guidance on facilitating group work and selecting and sequencing student work for use in whole-class discussion. Ask groups to report out and reconcile any significant differences in responses. You may also want to facilitate discussion regarding the following:

- Based upon their analysis of Σ in Problem 2, you might ask undergraduates to specifically discuss $\mathrm{A}(\theta + 2k\pi)$ or to compare $\mathrm{A}(\theta)$ with $\mathrm{A}(-\theta)$.
- Based upon their analysis of Φ in Problem 3, you might ask undergraduates to compare $\mathrm{B}(\theta)$ with $\mathrm{B}(\theta + \pi)$.

Next, write the refined (correct) geometric descriptions of the sets on the board for them to reference throughout the Class Activity. It may help to include a geometric interpretation of the parameter θ for each set. So:

- Σ represents the set of rotations about the origin by θ radians.
- Φ represents the set of reflections about lines through the origin found by traveling θ radians from the positive x-axis.

After writing this list on the board, discuss the following connection to teaching:

Discuss This Connection to Teaching

Rigid motions (translations, reflections, and rotations) of geometric figures are used in high school geometry to assess the congruence of shapes. Translating this work into the language of matrix operations allows prospective teachers the chance to see how intuitive geometric ideas can be represented algebraically to provide an additional method of inquiry for supporting conjectures and conclusions.

Based upon recommendations of the CCSSM (2010), there has been a shift to a transformation approach to the teaching of geometric concepts of congruence, similarity, and symmetry. Mathematics educators have been calling for this shift in approach since the 1900s (Lai & Donsig, 2018). The emphasis on defining similarity based upon geometric transformations may allow students to apply this mathematical concept in meaningful ways (Seago, et al., 2013) and possibly address Ada and Kurtuluş' (2010) finding that students often seemed to know the algebraic definitions of translation and rotation, but not able to attend to the conceptual or associated geometric meanings. Nonetheless, Seago, et al. (2013) explain that attending to the recommendations of the CCSSM poses serious challenges for supporting both teachers and students in shifting from a traditional, static approach. Thus, providing opportunities for encountering ways to revisit the transformation approach to geometry targets supporting undergraduates' conceptual progress in this regard. Further, Lai and Donsig (2018) propose that "teaching geometry from a transformational approach provides an opportunity to showcase abstract algebraic ideas in ways that are accessible and relevant to secondary mathematics" (p. 63).

The purpose of Problem 4 is to prepare or enable undergraduates to quickly address the associativity of the particular 2×2 matrix subsets in the Class Activity. It is helpful to demonstrate that often, a set will inherit associativity under a particular operation from a more familiar set in which it is contained.

Pre-Activity Problem 4

4. Recall that 2×2 matrix multiplication is associative. On the other hand, 2×2 matrices do not always commute under matrix multiplication. Give an example of a pair of 2×2 matrices that do commute and a pair of 2×2 matrices that do NOT commute under matrix multiplication.

Sample Response:
Any matrix commutes with the identity matrix. On the other hand,

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}$$

Commentary:
If you collected the Pre-Activity in advance, you might choose to showcase a relevant example or two from undergraduate submissions (such as a pair of diagonal matrices, which would be helpful for Class Activity Problem 3). Otherwise, only spend time reviewing this problem if their work reveals that review is needed.

Transition to the Class Activity by telling undergraduates that, now that we have identified these sets of matrices as particular geometric transformations, it is useful to explore the structure of these sets: we want to verify whether they are groups under multiplication so that we can use group theory as a tool to study them.

Class Activity: Problems 1 & 2 (20 minutes)

Distribute the Class Activity. Before undergraduates begin working on Problem 1 in small groups, remind them that they should not be computing the entries of any matrices or doing any matrix arithmetic—they should be "translating" the equations into an English sentence or a diagram, like the example provided in Problem 1(a). Allow the class to work on Problems 1(b) through 1(e) in small groups.

As you circulate the classroom and monitor undergraduates' responses, consider using any of the following questions to prompt further discussion:

- Besides $\theta = 2\pi$, what other values of θ give the identity matrix?
- Besides $-\theta$, what other angle measure could we use to "undo" a rotation by θ radians?
- Describe how the matrix product $\mathrm{A}(\theta_1)\mathrm{A}(\theta_2)$ will transform a vector v in $\mathbb{R}^2$. Which matrix acts first on v? Does this matter?

Class Activity Problem 1

1. Consider the set Σ given below.

$$\Sigma = \left\{ \mathrm{A}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} : \theta \in \mathbb{R} \right\}$$

Restate the following equations in terms of the geometric effect matrices in Σ have on vectors in $\mathbb{R}^2$. Do not make any calculations—your answers should be written sentences or labeled sketches. A sample solution to part (a) and a partial solution to part (b) are provided.

(a) $\mathrm{A}(\theta_1)[\mathrm{A}(\theta_2)\mathrm{A}(\theta_3)] = [\mathrm{A}(\theta_1)\mathrm{A}(\theta_2)]\mathrm{A}(\theta_3)$
When rotating a vector by three different angles, the way that I group the rotations before doing them does not affect the overall rotation.

(b) $\mathrm{A}(\theta_1)\mathrm{A}(\theta_2) = \mathrm{A}(\theta_1 + \theta_2)$
Solution:
Rotating a vector by θ_2 radians and then by θ_1 radians is equivalent to rotating it by $\theta_1 + \theta_2$ radians.

(c) $\mathrm{A}(\theta)^{-1} = \mathrm{A}(-\theta)$
Solution:
The inverse of the matrix representing a rotation of θ radians is the matrix corresponding to a rotation of -θ radians. That is, the matrix that rotates the same amount in the opposite direction.

(d) $\mathrm{A}(2\pi) = \mathrm{I}$
Solution:
The matrix representing a rotation of 2π radians is just the identity matrix.

(e) $\mathrm{A}(\theta_1)\mathrm{A}(\theta_2) = \mathrm{A}(\theta_2)\mathrm{A}(\theta_1)$
Solution:
When composing two rotations, either one can be applied first. That is, rotating by θ_1 and then θ_2 radians is no different than rotating by θ_2 and then θ_1 radians.

After groups share their answers with the class, askl them to work on Problem 2 and emphasize that they should not be doing any matrix calculations for this problem. As you monitor the groups, you may point out, when needed, that they have already addressed most of this problem on the Pre-Activity. Consider making the following points as needed:

- To justify that Σ is closed and commutative under matrix multiplication, encourage undergraduates to appeal to their geometric intuition from Pre-Activity Problems 1(a) and 1(b).
- For associativity, refer undergraduates to Pre-Activity Problem 4.
- For identity, undergraduates will be tempted to compute $\mathrm{A}(2\pi)$ to see that it is the 2×2 identity matrix. Make sure they also consider why $\mathrm{A}(2k\pi) = \mathrm{I}$ makes geometric sense.
- To find an inverse without calculations, encourage undergraduates to think about how they might "undo" a rotation. That is, how they could return a vector to its original position after it has been rotated by θ radians.

Class Activity Problem 2

2. Explain whether Σ is an abelian group under matrix multiplication by analyzing your responses to Problem 1. Does "order matter" when multiplying together a string of matrices from Σ?

Solution:

- 1(a) corresponds with associativity. As we saw in Problem 4 of the Pre-Activity, 2×2 matrix multiplication is associative, and because Σ is a subset of 2×2 matrices, it is associative with respect to matrix multiplication.
- 1(b) corresponds with closure, and appears true by inspection.
- 1(c) corresponds with the existence of inverses, and appears true by inspection.
- 1(d) corresponds with the existence of an identity element, and is clearly true when the entries of $\mathrm{A}(2\pi)$ are calculated. Alternatively, since 2π is a full rotation, it makes sense that $\mathrm{A}(2\pi)$ would not change the position of any vector on which it acts.
- 1(e) corresponds with commutativity. If one accepts that Σ is closed, we can apply $\mathrm{A}(\theta_1)\mathrm{A}(\theta_2) = \mathrm{A}(\theta_1 + \theta_2)$ and the commutativity of real numbers to demonstrate the result.

$\Rightarrow$ Σ appears to be a commutative group under matrix multiplication.

Convene a classroom discussion to establish that Σ is in fact a group. Discuss the following connection to teaching:

Discuss This Connection to Teaching

The ability to "undo" something is key to many high school mathematics topics (e.g., inverse functions, solving equations by using additive/multiplicative inverses, etc.) By emphasizing this aspect of algebraic structure, prospective teachers can appreciate how working within a group (eventually, a field) guarantees that certain familiar operations are appropriate and valid.

We would like undergraduates to think about these conditions geometrically, but there are algebraic alternatives for advanced classes (or if time permits):

- *For closure, undergraduates will need to use the angle sum formulas for sine and cosine.*
- *For commutativity, undergraduates should refer to the fact that they have proved* $\mathrm{A}(\theta_1)\mathrm{A}(\theta_2) = \mathrm{A}(\theta_1 + \theta_2)$ *and appeal to the commutativity of* θ_1 *and* θ_2 *as real numbers.*
- *For inverses, undergraduates can use the fact that they have now shown both* $\mathrm{A}(\theta_2)\mathrm{A}(\theta_1) = \mathrm{A}(\theta_1)\mathrm{A}(\theta_2) = \mathrm{A}(\theta_1 + \theta_2)$ *and* $\mathrm{A}(2k\pi) = \mathrm{I}$ *to construct a reasonable algebraic argument.*

Class Activity: Problems 3 & 4 (20 minutes)

Allow the class to work on Problem 3 in small groups. Before they begin, you may wish to remind undergraduates that matrices in this set represent reflections about a line through the origin (and that line is determined by the angle θ).

Class Activity Problem 3 : Parts a, b, & c

3. Consider the set Φ given below.

$$\Phi = \left\{ \mathrm{B}(\theta) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} : \theta \in \mathbb{R} \right\}$$

Answer the following questions by considering the geometric effect matrices in Φ have on vectors in $\mathbb{R}^2$. Do not make any calculations—your answers should be written sentences or labeled sketches.

(a) Let $\mathrm{B}(\theta_1)$ be a particular matrix in Φ. Do you think $\mathrm{B}(\theta_1)$ has a multiplicative inverse in Φ? That is, is there a value θ_2 for which $\mathrm{B}(\theta_2) = \mathrm{B}(\theta_1)^{-1}$? Explain why or why not.

Solution:
Yes, and $\theta_2 = \theta_1$. If you have reflected a vector over a line, to return it to its original position you can always reflect it over the same line; that is, each matrix in Φ is its own inverse.

(b) Does Φ contain the identity matrix? That is, is there a value θ for which $\mathrm{B}(\theta) = I$? Explain why or why not.

Solution:
Reflecting a vector about a line through the origin will almost always produce a different vector (the exception being when the vector coincides with the line of reflection). The identity matrix cannot be in Φ.

(c) Explain how your answers to Problems 3(a) and 3(b) can be used to determine that Φ is NOT closed under matrix multiplication.

Solution:
We see that $\mathrm{B}(\theta_1)\mathrm{B}(\theta_1) = \mathrm{I}$ by Problem 3(a), but I is not in Φ by Problem 3(b). Thus, there exists a pair of matrices in Φ whose product is not again in Φ.

Commentary:
As you circulate the classroom:

- For Problem 3(a), remind undergraduates to think geometrically: the inverse matrix of $\mathrm{B}(\theta_1)$ is the matrix in Φ which "undoes" the reflection given by $\mathrm{B}(\theta_1)$.
- For Problem 3(b), undergraduates may recognize that I is not in Φ by reasoning about the signs of the entries along the diagonal of $\mathrm{B}(\theta)$. If so, encourage them to also provide a geometric explanation of why no matrix in Φ could represent the identity transformation.
- For Problem 3(b), you might also prompt undergraduates to explain why it is not contradictory for it to be true that $\mathrm{B}(2\pi)(1,0) = (1,0)$ even though we know that $\mathrm{B}(2\pi) \neq \mathrm{I}$. This type of reasoning is important for helping undergraduates make sense of quantifiers in the group axioms.

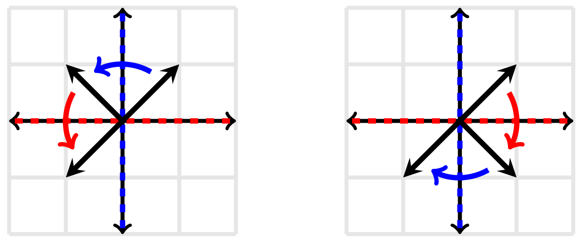
After most groups have completed Problems 3(a)–3(c), check in as a whole class to make sure everyone is on the same page. Discuss the following connection to teaching to set the stage before continuing on to Problem 3(d).

Discuss This Connection to Teaching

Problems such as Problem 3(d) are valuable because it's useful for everyone to think about how others use and reason with mathematics. It also gives prospective teachers or undergraduates who are considering work as a tutor or teaching assistant an opportunity to think about how they would respond to student work in ways that help students develop an understanding of the concept. Generating multiple questions for Luisa models that teachers often need to have several different questions prepared to facilitate different ways students might be thinking about the problem.

Class Activity Problem 3 : Part d

(d) Luisa sketches two diagrams, pictured below, to illustrate two different orders in which an arbitrary vector might be reflected over both the x and y axes.



Luisa then verifies arithmetically that $\mathrm{B}(\frac{\pi}{2})\mathrm{B}(\pi) = \mathrm{B}(\pi)\mathrm{B}(\frac{\pi}{2})$. Based on the diagrams and her supporting calculations, she concludes that elements of Φ commute under matrix multiplication.

i. Luisa's conclusion is incorrect. What understanding has Luisa demonstrated in her work? What has she proven?

Sample Responses:

- Luisa has demonstrated that she knows the matrices of the set Φ represent reflections over a line through the origin.
- Luisa understands matrix multiplication and how to graph vectors as they are transformed by a matrix.
- Luisa knows the definition of commutativity.
- Luisa has only proven only that the two matrices she has chosen, representing two specific reflections from the set Φ, commute with each other.

ii. Explain the error in Luisa's reasoning.

Sample Response:

Luisa has shown that a single pair of matrices in Φ commute, but because she did not choose two arbitrary matrices, she has not shown that ALL possible pairs of matrices in Φ commute.

iii. Give two questions you could ask Luisa to help her understand her error. Explain why your questions are helpful.

Sample Undergraduate Response(s):

- What does it mean for an entire set to be commutative under an operation? This question will help Luisa think about the quantifiers in the commutativity requirement for abelian groups.
- I want to prove that all horses are brown and bring you one brown horse. Have I convinced you? This question will help Luisa think about the quantifiers in the commutativity requirement for abelian groups.
- If we calculate the entries of $\mathrm{B}(\frac{\pi}{2})$ and $\mathrm{B}(\pi)$, what special property do we see these matrices have? Do all matrices in this set have this property? These questions will help Luisa think about the commutativity of diagonal and non-diagonal matrices, in particular whether reflections are always diagonal.

Give the class only a few minutes to consult their neighbors about Problem 4—their work on Problem 3 should have provided much of the insight for addressing this problem. As you monitor their work, select groups that will report our their responses. After an appropriate sequence of presentations from the groups, move on to the next part of the Class Activity.

Class Activity Problem 4

4. Explain whether Φ is a group under matrix multiplication by analyzing your responses to Problem 3. Does "order matter" when multiplying together a string of matrices from Φ?

Solution:

- Φ is not a group; by Problem 3(c) it is not closed, and by Problem 3(b) it does not contain an identity element.
- Order does matter when applying reflections; by Problem 3(d), we see that reflections do not always commute.

Class Activity: Problems 5 & 6 (20 minutes)

Allow undergraduates to work in small groups on Problem 5. As you monitor and facilitate their group work, consider the following prompts to encourage discussion. These prompts may also be posed for class discussion.

- How can we interpret this activity in terms of matrices acting on vectors in $\mathbb{R}^2$? Can you write Todd's steps using matrices from Σ and Φ?
- Can you find a single transformation, rather than a sequence, that will move triangle F back to its original position? What does this imply about the composition of a rotation and a reflection?

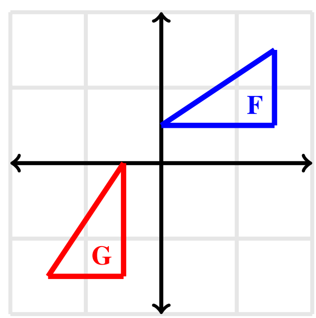
Class Activity Problem 5

5. Todd, a high school geometry student, is attempting to show that the two triangles pictured to the right are congruent. To do so, he must use some combination of reflections and rotations to move triangle F on top of triangle G. Todd concludes that he should:

   - Reflect F over the y-axis.
   - Rotate F counterclockwise 90° about the origin.

   To move F back to its original position, Todd says he can make these same two transformations in reverse order. That is, once F has been moved to the same position as G, he would:

   - Rotate F counterclockwise 90° about the origin.
   - Reflect F over the y-axis.



   (a) Why might Todd expect this procedure to work?

   Sample Responses:

   - Todd might not expect order to matter when applying transformations, especially if he has not yet learned about commutativity.
   - Todd might be thinking that if you reverse the order that you transform something, it will have the opposite effect.

   (b) Explain the error in Todd's reasoning.

   Sample Responses:

   - Todd's first step is okay, but next he needs to reflect over the x-axis instead of the y-axis.
   - Todd isn't thinking about the inverse of each individual transformation. He's just doing the same thing in the opposite order.

   (c) Find a sequence of transformations that will move F back to its original position. Explain, using vocabulary or notation from this course, how you know your steps are correct.

Solution(s):

- First, rotate F clockwise 90° about the origin. Then, reflect F over the y-axis. This will return F back to its original position because $[\mathrm{A}(\frac{\pi}{2})\mathrm{B}(\frac{\pi}{2})]^{-1} = \mathrm{B}(\frac{\pi}{2})^{-1}\mathrm{A}(\frac{\pi}{2})^{-1} = \mathrm{B}(\frac{\pi}{2})\mathrm{A}(\frac{-\pi}{2})$.
- You can also repeat the same sequence of transformations in the same order.

Problem 6 can be posed and answered as a class without any preceding group work.

Class Activity Problem 6

6. It turns out that the set of all rotations and reflections, $\Sigma \cup \Phi$, is itself a group under multiplication (you do not need to prove this). Does "order matter" when multiplying together a string of matrices from $\Sigma \cup \Phi$?

Solution:
Because $\Sigma \cup \Phi$ contains the reflections as a subset and we know that order matters for reflections, order also matters, in general, for $\Sigma \cup \Phi$.

Tie Problems 5 and 6 to the work of teaching by discussing the following connection:

Discuss This Connection to Teaching

Rigid motions are used in high school geometry to justify the congruence of geometric figures. Problems 5 and 6 require undergraduates to translate between high school geometry concepts and advanced mathematical language. This process emphasizes to prospective teachers that knowledge of matrices and underlying group structure can inform solutions in their future classrooms; in this case, a sequence of rigid motions can always be "undone" because of the underlying group structure, but that order does in fact matter when considering a sequence of rigid motions.

Wrap-Up (5 minutes)

Recap the lesson briefly for the class:

- We can represent particular rigid motions as sets of matrices and consider whether they are groups under matrix multiplication:
  - The set of rotations about the origin are an abelian group.
  - The set of reflections about lines through the origin are not a group, nor do they always commute.
  - The union of the two sets above is a group, but not an abelian one.
- Group structure can help indicate some situations in which "order doesn't matter": because rotations are an abelian group, we can compose those transformations however we like. We must be more careful when dealing with reflections.

We conclude the lesson by using an exit ticket. See Chapter 1 for guidance on how to conclude mathematics lessons using exit tickets.

Homework Problems

At the end of the lesson, assign the following homework problems.

Problem 1 prompts undergraduates to interpret another student's thinking in a way that allows them to generate a pair of guiding questions. In doing so, they practice interpreting the order of quantifiers in the identity group axiom; multiply quantified statements are a difficult topic for some undergraduates.

Homework Problem 1

1. Recall that Σ represents the set of rotations about the origin and that Φ represents the set of reflections across lines through the origin. These sets are given below:

$$\Sigma = \left\{ \mathrm{A}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} : \theta \in \mathbb{R} \right\} \quad \Phi = \left\{ \mathrm{B}(\theta) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} : \theta \in \mathbb{R} \right\}$$

Reasoning geometrically, Jordan finds that $\mathrm{B}(\frac{\pi}{4})v = v$ for any vector v whose vertical and horizontal components are equal.

(a) What geometric understanding might Jordan have of the set Φ that enabled them to draw this conclusion without making any computations?

Sample Responses:

- Jordan recognized that Φ was the set of reflections about lines through the origin.
- Jordan knows that the line of reflection used by $\mathrm{B}(\frac{\pi}{4})$ to transform a vector lies $\frac{\pi}{4}$ radians above the positive x-axis.
- Jordan knows that the vector v must form an angle of either $\frac{\pi}{4}$ or $\frac{5\pi}{4}$ radians with the positive x-axis. In either case, v will coincide with the line over which $\mathrm{B}(\frac{\pi}{4})$ reflects and v will not be transformed.

(b) Jordan claims that the work above shows that $\mathrm{B}(\frac{\pi}{4})$ is the identity matrix. Explain the error in Jordan's reasoning.

Solution:

Jordan does not seem to understand that the identity matrix I must satisfy $\mathrm{I}v = v$ for ALL vectors $v \in \mathbb{R}^2$. This matrix only functions as an identity for vectors of the form (a, a).

(c) Give two questions you could ask Jordan to help them understand their error. Why would your questions be helpful?

Sample Responses:

- Does $\mathrm{B}(\frac{\pi}{4})$ act as the identity on $w = (1, 0)$? This question would be helpful because Jordan might not realize $\mathrm{B}(\frac{\pi}{4})$ is only the identity matrix if $\mathrm{B}(\frac{\pi}{4})v = v$ for ALL $v \in \mathbb{R}^2$.
- What are the entries of $\mathrm{B}(\frac{\pi}{4})$? This question would be helpful because it will show Jordan that the diagonal entries of any matrix $\mathrm{B}(\theta)$ cannot both be 1.

Undergraduates should understand that binary operations need not be "nice," i.e., only consisting of some combination of the usual, familiar operations. Problem 2 introduces undergraduates to an unconventional binary operation that is in fact well-behaved: order doesn't matter when applying $\diamond$.

Homework Problem 2

2. Consider the operation $\diamond$ given by $a \diamond b = a^{\log(b)}$ on the set of positive real numbers, $\mathbb{R}^+$.

(a) Is $\diamond$ closed on this set? If so, justify your conclusion. If not, provide a specific example of $a, b \in \mathbb{R}^+$ for which $a \diamond b \notin \mathbb{R}^+$.

Solution:

The operation $\diamond$ is closed since, $\forall\, a, b \in \mathbb{R}^+$, we have $\log(b) \in \mathbb{R}$, from which it follows that $a^{\log(b)} \in \mathbb{R}^+$.

(b) Is $\diamond$ an associative operation on this set? If so, justify your conclusion. If not, provide a specific example of $a, b, c \in \mathbb{R}^+$ for which $a \diamond (b \diamond c) \neq (a \diamond b) \diamond c$.

Solution:
The operation $\diamond$ is associative. Below, we apply the fact that $a \diamond b = a^{\log(b)} = 10^{\log(a)\log(b)}$:
$a \diamond (b \diamond c) = a^{\log[10^{\log(b)\log(c)}]} = a^{\log(b)\log(c)} = 10^{\log(a)\log(b)\log(c)}$
$(a \diamond b) \diamond c = [10^{\log(a)\log(b)}]^{\log(c)} = 10^{\log(a)\log(b)\log(c)}$

(c) Is $\diamond$ a commutative operation on this set? If so, justify your conclusion. If not, provide a specific example of $a, b \in \mathbb{R}^+$ for which $a \diamond b \neq b \diamond a$.

Solution:
The operation is $\diamond$ commutative: $a \diamond b = a^{\log(b)} = [10^{\log(a)}]^{\log(b)} = [10^{\log(b)}]^{\log(a)} = b^{\log(a)} = b \diamond a$

(d) Does "order matter" under this operation? Explain why or why not.

Solution:
No. The operation $\diamond$ is clearly closed, and we have demonstrated that it is also both commutative and associative.

The impression that commutativity is "stronger" than associativity sometimes leads to the misconception that the former implies the latter. Averaging is a binary operation (familiar even to high school students such as Aisling) that is commutative but non-associative, providing an interesting example of when order does matter that is not likely already a part of undergraduates' concept image. Problem 3 gives undergraduates the opportunity to leverage their understanding of binary operations and their properties in a teaching application.

Homework Problem 3

3. Aisling, a high school student, has made an 84 and a 72 on her first two precalculus assignments. She calculates her average in the course to be a 78. The following day, she receives a 90 on her next assignment. She makes the following calculation to compute her new average:

$$\frac{1}{2}(78 + 90) = 84$$

(a) What error has Aisling made?

Solution:
Averaging is not an associative operation; instead of averaging her grades one at a time, she should take the average all at once by adding and dividing by 3.

(b) Show that the operation $*$, given by $a * b = \frac{1}{2}(a + b)$ where $a, b \in \mathbb{R}^+$, is commutative. Does "order matter" under this operation? Explain why or why not.

Solution:
The operation is $*$ commutative: $a * b = \frac{1}{2}(a + b) = \frac{1}{2}(b + a) = b * a$. However, "order matters" because $*$ is not associative.

(c) Consider the following questions that you might ask Aisling:

i. Explain why the question below might not help Aisling:

Should your average be lower than 84?

Sample Response:
Aisling doesn't know what her average should be; that is why she was calculating it. A teacher might be able to average three grades in their head pretty easily, but a student probably can't. This doesn't help Aisling see what she did wrong, it just makes her think her answer isn't right.

ii. Explain how the question below might help you advance Aisling's understanding:

What would your average be if you had made a 90, then a 72, then an 84?

Sample Response:
By asking Aisling to recalculate her average in a different order, she will probably get a different answer than her first calculation. Hopefully, she will see that "order matters" when computing an average—that is, you can't just average a bunch of numbers one after another and need to find a way to do it all at once.

Problem 4 formally establishes that we do not need to check that an element is both a left and a right inverse (as long as the operation in question is associative), and thus that a definition for a group that at first glance appears "weaker" is the same.

Homework Problem 4

4. Let G be a set with associative operation $*$ and with identity element e. Assume that every element of G has a left inverse: that is, $\forall\, a \in G, \exists\, b \in G$ such that $b * a = e$.

   (a) Show that b must also be a right inverse of a: that is, we also have $a * b = e$.

   Solution:
   Since $b \in G$, it must also be true that b has a left inverse c. That is, $\exists\, c \in G$ such that $c * b = e$. Then: $b * a = e \Rightarrow (b * a) * b = b \Rightarrow c * ((b * a) * b) = c * b \Rightarrow e * (a * b) = e \Rightarrow a * b = e$, as desired.

   (b) Explain how the associativity of $*$ plays a key role in your proof for 4(a).

   Solution:
   In the above proof, we assert that $(b * a) * b = b \Rightarrow c * ((b * a) * b) = c * b \Rightarrow e * (a * b) = e$. This is only true because associativity allows us to rewrite the expression $c * ((b * a) * b)$ as $(c * b) * (a * b)$, allowing us to apply the fact that c is the left inverse of b.

   (c) Examine a list of axioms that you've seen presented in the definition of a group. How does your work in this problem affect your understanding of these axioms?

   Sample Response:
   The group axiom which states that $\forall\, g \in G, \exists\, g^{-1} \in G$ such that $g * g^{-1} = g^{-1} * g = e$ is stronger than necessary. We need only verify that every element has a left inverse, that is, $\forall\, g \in G$, $\exists\, g^{-1} \in G$ such that $g^{-1} * g = e$. The fact that this left inverse is also the right inverse follows directly. Alternatively, the same proof as above slightly modified suffices to show that a right inverses are also left inverse. Essentially, group elements always commute with their inverse element. All of the above commentary requires that the set in question is already associative, however.

Translations are an important category of rigid motion that undergraduates may have used in high school geometry. Undergraduates might assume that translations can also be represented as matrices after seeing the such a treatment of the other rigid motions (reflections and rotations) in this lesson. In Problem 5 we justify that this is not true.

Homework Problem 5

5. We have encountered matrices which represent rotations and reflections of vectors in $\mathbb{R}^2$. Does there exist a 2×2 matrix which represents the translation of vectors? If so, write it down and justify how you know it represents translations. If not, explain.

   Solution:
   No. Matrices represent linear transformations, so if there existed a matrix A that was a translation, we would require $A\vec{0} = \vec{0}$. However, if A is any non-identity translation, this would not be the case.

Undergraduates are first given the opportunity to prove an important result algebraically that adds depth to results observed during the Class Activity. Problem 6 continues the trend of framing meaningful results about group structure using geometric language.

Homework Problem 6

6. Show that the product of any two reflection matrices is a rotation matrix. [Hint: You will need the angle subtraction formulas for sine and cosine]. Using this result, give a geometric description of when two reflection matrices will commute.

Solution:

$$\begin{aligned} B(\theta_1)B(\theta_2) &= \begin{bmatrix} \cos(2\theta_1) & \sin(2\theta_1) \\ \sin(2\theta_1) & -\cos(2\theta_1) \end{bmatrix} \begin{bmatrix} \cos(2\theta_2) & \sin(2\theta_2) \\ \sin(2\theta_2) & -\cos(2\theta_2) \end{bmatrix} \\ &= \begin{bmatrix} \cos(2\theta_1)\cos(2\theta_2) + \sin(2\theta_1)\sin(2\theta_2) & \sin(2\theta_2)\cos(2\theta_1) - \sin(2\theta_1)\cos(2\theta_2) \\ \sin(2\theta_1)\cos(2\theta_2) - \sin(2\theta_2)\cos(2\theta_1) & \cos(2\theta_1)\cos(2\theta_2) + \sin(2\theta_1)\sin(2\theta_2) \end{bmatrix} \\ &= \begin{bmatrix} \cos(2(\theta_1-\theta_2)) & \sin(2(\theta_2-\theta_1)) \\ \sin(2(\theta_1-\theta_2)) & \cos(2(\theta_1-\theta_2)) \end{bmatrix} \\ &= \begin{bmatrix} \cos(2(\theta_1-\theta_2)) & -\sin(2(\theta_1-\theta_2)) \\ \sin(2(\theta_1-\theta_2)) & \cos(2(\theta_1-\theta_2)) \end{bmatrix} \\ &= A(2(\theta_1-\theta_2)) \end{aligned}$$

Where we use the fact that sine is an odd function when moving from the third to the fourth line. By this result, $B(\theta_1)B(\theta_2) = B(\theta_2)B(\theta_1)$ when $A(2(\theta_1-\theta_2)) = A(2(\theta_2-\theta_1))$, or equivalently when $2(\theta_1-\theta_2) = 2(\theta_2-\theta_1) + 2k\pi$ for some $k \in \mathbb{Z}$. Simplifying, we see that two reflection matrices commute only when $\theta_1 = \theta_2 + k\frac{\pi}{2}$. That is, two reflection matrices will commute when the lines over which they reflect are either orthogonal or identical.

Assessment Problems

The following two problems address ideas explored in the lesson, with a focus on connections to teaching and mathematical content. You can include these problems as part of your usual course quizzes or exams.

Assessment Problems 1 & 2

1. Consider the set Δ, given below.

$$\Delta = \left\{ \mathrm{C}(a) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} : a \in \mathbb{R}^+ \right\}$$

   (a) Describe the geometric effect that the matrix $\mathrm{C}(a)$ has on a vector $v \in \mathbb{R}^2$.

   Solution:
   The set Δ represents the dilations. That is, the matrix $\mathrm{C}(a)$ stretches or compresses a vector by a factor of a, depending on whether a is greater or less than 1, respectively.

(b) Is Δ an abelian group under matrix multiplication? Demonstrate why or why not.

Solution:

- The set is closed under multiplication: $\mathrm{C}(a)\mathrm{C}(b) = \mathrm{C}(ab)$, and $ab \in \mathbb{R}^+$ since both a and b are positive real numbers.
- The set of all 2×2 matrices is associative, so Δ, as a subset, must also be associative.
- It is clear that $\mathrm{C}(1) = \mathrm{I}$, so there is a group identity.
- If $a \in \mathbb{R}^+$, then $\frac{1}{a} \in \mathbb{R}^+$. Also: $\mathrm{C}(a)\mathrm{C}(\frac{1}{a}) = \mathrm{C}(\frac{a}{a}) = \mathrm{C}(1) = \mathrm{I}$, so each element has a group inverse.
- Finally, $\mathrm{C}(a)\mathrm{C}(b) = \mathrm{C}(ab) = \mathrm{C}(ba) = \mathrm{C}(b)\mathrm{C}(a)$, so Δ is abelian.

Thus, Δ is an abelian group, as desired.

2. Serena is working with the set Σ of rotations about the origin, given below.

$$\Sigma = \left\{ \mathrm{A}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} : \theta \in \mathbb{R} \right\}$$

She knows Σ is a group under multiplication. Reasoning geometrically, Serena argues that the matrices $A(-\frac{\pi}{4})$ and $A(\frac{7\pi}{4})$ both act as inverses of matrix $A(\frac{\pi}{4})$.

(a) What geometric understanding might Serena have of rotations about the origin that enabled her to draw this conclusion without making any computations?

Sample Responses:

- Serena knows that positive values of θ represent counterclockwise rotations and negative values of θ represent clockwise rotations.
- Serena recognizes that the inverse of $A(\frac{\pi}{4})$ could be conceptualized as either the matrix that rotates the opposite direction with equal magnitude, $A(-\frac{\pi}{4})$, or the matrix which rotates the rest of the way around the circle in the same direction, i.e. $A(2\pi - \frac{\pi}{4}) = A(\frac{7\pi}{4})$. Both of these matrices would return a vector to its original position after being transformed by $A(\frac{\pi}{4})$.

(b) Serena claims that her work above shows that the group of rotations is a counterexample to the claim that all group elements have a unique inverse. Explain the error in Serena's reasoning.

Sample Response:

She assumes that having differing θ values leads to unique matrices in A when really it's based on the evaluated $\sin(\theta)$ and $\cos(\theta)$ values.

(c) What question would you ask Serena to help her understand her error? Why is your question helpful?

Sample Responses:

- What are the matrix representations of $A(-\frac{\pi}{4})$ and $A(\frac{7\pi}{4})$? This question would be helpful to show that both angles would produce the same matrix.
- Where on the unit circle can you find $-\frac{\pi}{4}$ and $\frac{7\pi}{4}$? This question would be helpful to show that they are coterminal angles, so they have the same values for sine and cosine.
- Are the elements of the group real numbers or matrices? This question would help Serena realize that the parameters might not be the same, but that it would not matter if the resultant matrices are identical.

1.6 References

[1] Ada, T., & Kurtuluş, A. (2010). Students' misconceptions and errors in transformation geometry. *International Journal of Mathematical Education in Science and Technology*, *41*(7), 901-909.

[2] Lai, Y., & Donsig, A. (2018). Using geometric habits of mind to connect geometry from a transformation perspective to graph transformations and abstract algebra. *Connecting abstract algebra to secondary mathematics, for secondary mathematics teachers*, 263–289.

[3] National Governors Association Center for Best Practices & Council of Chief State School Officers (2010). *Common Core State Standards for Mathematics.* Authors. Retrieved from `http://www.corestandards.org/`

[4] Seago, N., Jacobs, J., Driscoll, M., Nikula, J., Matassa, M., & Callahan, P. (2013). Developing teachers' knowledge of a transformations-based approach to geometric similarity. *Mathematics Teacher Educator*, *2*(1), 74-85.

1.7 Lesson Handouts

Handouts for use during instruction are included on the pages that follow. LaTeX files for these handouts can be downloaded from INSERT URL HERE.

NAME: **PRE-ACTIVITY:** GROUPS OF TRANSFORMATIONS (page 1 of 4)

Consider the sum $2+6+4+7$. When people say that "order does not matter" when computing such a sum they actually mean two things: the order of the individual terms of the sum can be rearranged without affecting the final result (for instance, $7+4+6+2$ and the original sum are sure to each give the same answer, 19) and, moreover, the order in which one chooses to compute the individual addition operations is unimportant (for instance, $((2+6)+4)+7$ and $2+((6+4)+7)$ both yield the same final result of 19). This conclusion relies on the three fundamental beliefs of integer arithmetic:

- Integer addition is **closed**; that is, $a+b$ is itself an integer for all integers a and b.
- Integer addition is **commutative**; that is, $a+b=b+a$ for all integers a and b.
- Integer addition is **associative**; that is, $(a+b)+c=a+(b+c)$ for all integers a, b, and c.

1. Consider the set of all rotations about the origin of the plane.
   [Recall that transformations (e.g., rotations) are functions. As such, for rotations r_α and r_β on $\mathbb{R}^2$, the *composition* of r_α followed by r_β, $r_\beta \circ r_\alpha$ is defined by $r_\beta \circ r_\alpha(P) = r_\beta(r_\alpha(P))$ where $P \in \mathbb{R}^2$.]

   (a) Is this set closed under composition? Explain.

   (b) Do rotations commute with each other under composition? Explain.

   (c) Do rotations about the origin satisfy the associative law under composition? Explain.

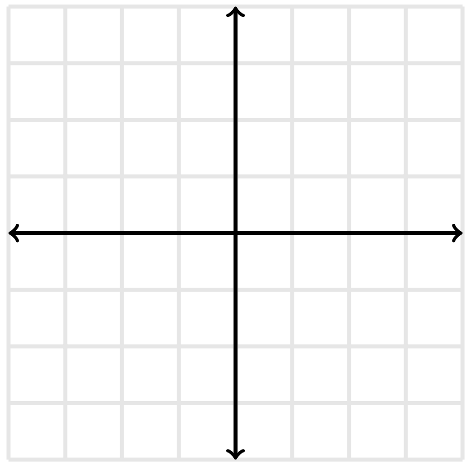
   (d) Does "order matter" when performing a series of rotations about the origin in the plane? Explain.

PRE-ACTIVITY: GROUPS OF TRANSFORMATIONS (page 2 of 4)

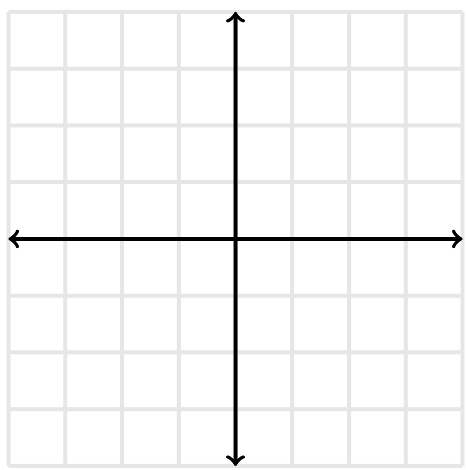
2. Consider the set Σ, given below.

$$\Sigma = \left\{ \mathrm{A}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} : \theta \in \mathbb{R} \right\}$$

(a) Calculate $A(\frac{\pi}{2})$. Choose three nonzero vectors v_1, v_2, and v_3 in $\mathbb{R}^2$ that are not all scalar multiples of one another. Compute $A(\frac{\pi}{2})v_1$, $A(\frac{\pi}{2})v_2$, and $A(\frac{\pi}{2})v_3$. Sketch all six vectors on the same coordinate plane.



(b) Repeat the process of 2(a) with $A(\theta)$ for a different nonzero value of θ and the same vectors.

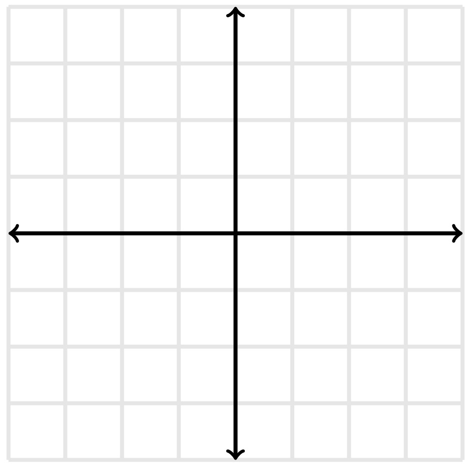


(c) Write a geometric description of how an arbitrary matrix from Σ acts on vectors in $\mathbb{R}^2$ based on your sketches in Problems 2(a) and 2(b).

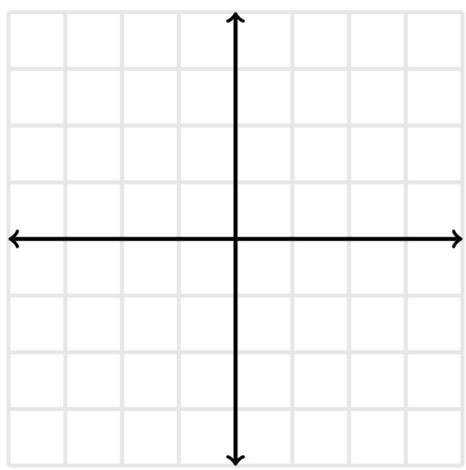
3. Consider the set Φ, given below.

$$\Phi = \left\{ \mathrm{B}(\theta) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} : \theta \in \mathbb{R} \right\}$$

(a) Repeat the process of Problem 2(a) with matrix $B(\frac{\pi}{2})$.



(b) Repeat the process of Problem 3(a) with $B(\theta)$ for a different value of θ and the same vectors.



(c) Write a geometric description of how an arbitrary matrix from Φ acts on vectors in $\mathbb{R}^2$ based on your sketches in Problems 3(a) and 3(b).

4. Recall that 2×2 matrix multiplication is associative. On the other hand, 2×2 matrices do not always commute under matrix multiplication. Give an example of a pair of 2×2 matrices that do commute and a pair of 2×2 matrices that do NOT commute under matrix multiplication.

NAME: **CLASS ACTIVITY:** GROUPS OF TRANSFORMATIONS (page 1 of 4)

1. Consider the set Σ given below.

$$\Sigma = \left\{ \mathrm{A}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} : \theta \in \mathbb{R} \right\}$$

Restate the following equations in terms of the geometric effect matrices in Σ have on vectors in $\mathbb{R}^2$. Do not make any calculations—your answers should be written sentences or labeled sketches. A sample solution to part (a) and a partial solution to part (b) are provided.

(a) $\mathrm{A}(\theta_1)[\mathrm{A}(\theta_2)\mathrm{A}(\theta_3)] = [\mathrm{A}(\theta_1)\mathrm{A}(\theta_2)]\mathrm{A}(\theta_3)$

When rotating a vector by three different angles, the way that I group the rotations before doing them does not affect the overall rotation.

(b) $\mathrm{A}(\theta_1)\mathrm{A}(\theta_2) = \mathrm{A}(\theta_1 + \theta_2)$

Rotating a vector by _____ radians and then by _____ radians is equivalent to rotating it by ___________ radians.

(c) $\mathrm{A}(\theta)^{-1} = \mathrm{A}(-\theta)$

(d) $\mathrm{A}(2\pi) = \mathrm{I}$

(e) $\mathrm{A}(\theta_1)\mathrm{A}(\theta_2) = \mathrm{A}(\theta_2)\mathrm{A}(\theta_1)$

2. Explain whether Σ is an abelian group under matrix multiplication by analyzing your responses to Problem 1. Does "order matter" when multiplying together a string of matrices from Σ?

CLASS ACTIVITY: GROUPS OF TRANSFORMATIONS (page 2 of 4)

3. Consider the set Φ given below.

$$\Phi = \left\{ \mathrm{B}(\theta) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} : \theta \in \mathbb{R} \right\}$$

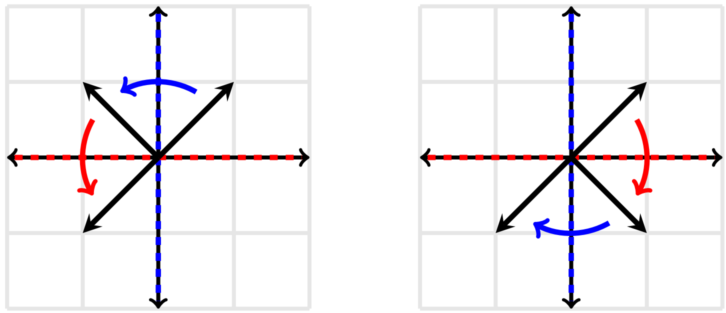
Answer the following questions by considering the geometric effect matrices in Φ have on vectors in $\mathbb{R}^2$. Do not make any calculations—your answers should be written sentences or labeled sketches.

(a) Let $\mathrm{B}(\theta_1)$ be a particular matrix in Φ. Do you think $\mathrm{B}(\theta_1)$ has a multiplicative inverse in Φ? That is, is there a value θ_2 for which $\mathrm{B}(\theta_2) = \mathrm{B}(\theta_1)^{-1}$? Explain why or why not.

(b) Does Φ contain the identity matrix? That is, is there a value θ for which $\mathrm{B}(\theta) = I$? Explain why or why not.

(c) Explain how your answers to Problems 3(a) and 3(b) can be used to determine that Φ is NOT closed under matrix multiplication.

(d) Luisa sketches two diagrams, pictured below, to illustrate two different orders in which an arbitrary vector might be reflected over both the x and y axes.



Luisa then verifies arithmetically that $\mathrm{B}(\frac{\pi}{2})\mathrm{B}(\pi) = \mathrm{B}(\pi)\mathrm{B}(\frac{\pi}{2})$. Based on the diagrams and her supporting calculations, she concludes that elements of Φ commute under matrix multiplication.

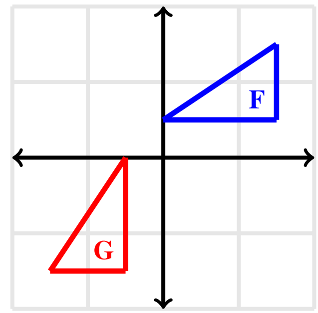
i. Luisa's conclusion is incorrect. What understanding has Luisa demonstrated in her work? What has she proven?

ii. Explain the error in Luisa's reasoning.

iii. Give two questions you could ask Luisa to help her understand her error. Explain why your questions are helpful.

4. Explain whether Φ is a group under matrix multiplication by analyzing your responses to Problem 3. Does "order matter" when multiplying together a string of matrices from Φ?

5. Todd, a high school geometry student, is attempting to show that the two triangles pictured to the right are congruent. To do so, he must use some combination of reflections and rotations to move triangle F on top of triangle G. Todd concludes that he should:



   - Reflect F over the y-axis.
   - Rotate F counterclockwise $90°$ about the origin.

   To move F back to its original position, Todd says he can make these same two transformations in reverse order. That is, once F has been moved to the same position as G, he would:

   - Rotate F counterclockwise $90°$ about the origin.
   - Reflect F over the y-axis.

   (a) Why might Todd expect this procedure to work?

   (b) Explain the error in Todd's reasoning.

   (c) Find a sequence of transformations that will move F back to its original position. Explain, using vocabulary or notation from this course, how you know your steps are correct.

6. It turns out that the set of all rotations and reflections, $\Sigma \cup \Phi$, is itself a group under multiplication (you do not need to prove this). Does "order matter" when multiplying together a string of matrices from $\Sigma \cup \Phi$?

NAME: **HOMEWORK PROBLEMS:** GROUPS OF TRANSFORMATIONS (page 1 of 2)

1. Recall that Σ represents the set of rotations about the origin and that Φ represents the set of reflections across lines through the origin. These sets are given below:

$$\Sigma = \left\{ \mathrm{A}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} : \theta \in \mathbb{R} \right\} \qquad \Phi = \left\{ \mathrm{B}(\theta) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} : \theta \in \mathbb{R} \right\}$$

   Reasoning geometrically, Jordan finds that $\mathrm{B}(\frac{\pi}{4})v = v$ for any vector v whose vertical and horizontal components are equal.

   (a) What geometric understanding might Jordan have of the set Φ that enabled them to draw this conclusion without making any computations?

   (b) Jordan claims that the work above shows that $\mathrm{B}(\frac{\pi}{4})$ is the identity matrix. Explain the error in Jordan's reasoning.

   (c) Give two questions you could ask Jordan to help them understand their error. Why would your questions be helpful?

2. Consider the operation $\diamond$ given by $a \diamond b = a^{\log(b)}$ on the set of positive real numbers, $\mathbb{R}^+$.

   (a) Is $\diamond$ closed on this set? If so, justify your conclusion. If not, provide a specific example of $a, b \in \mathbb{R}^+$ for which $a \diamond b \notin \mathbb{R}^+$.

   (b) Is $\diamond$ an associative operation on this set? If so, justify your conclusion. If not, provide a specific example of $a, b, c \in \mathbb{R}^+$ for which $a \diamond (b \diamond c) \neq (a \diamond b) \diamond c$.

   (c) Is $\diamond$ a commutative operation on this set? If so, justify your conclusion. If not, provide a specific example of $a, b \in \mathbb{R}^+$ for which $a \diamond b \neq b \diamond a$.

   (d) Does "order matter" under this operation? Explain why or why not.

3. Aisling, a high school student, has made an 84 and a 72 on her first two precalculus assignments. She calculates her average in the course to be a 78. The following day, she receives a 90 on her next assignment. She makes the following calculation to compute her new average:

$$\tfrac{1}{2}(78 + 90) = 84$$

   (a) What error has Aisling made?

   (b) Show that the operation $*$, given by $a * b = \frac{1}{2}(a + b)$ where $a, b \in \mathbb{R}^+$, is commutative. Does "order matter" under this operation? Explain why or why not.

   (c) Consider the following questions that you might ask Aisling:

      i. Explain why the question below might not help Aisling:

         Should your average be lower than 84?

      ii. Explain how the question below might help you advance Aisling's understanding:

         What would your average be if you had made a 90, then a 72, then an 84?

4. Let G be a set with associative operation $*$ and with identity element e. Assume that every element of G has a left inverse: that is, $\forall\, a \in G, \exists\, b \in G$ such that $b * a = e$.

   (a) Show that b must also be a right inverse of a: that is, we also have $a * b = e$.

   (b) Explain how the associativity of $*$ plays a key role in your proof for 4(a).

   (c) Examine a list of axioms that you've seen presented in the definition of a group. How does your work in this problem affect your understanding of these axioms?

5. We have encountered matrices which represent rotations and reflections of vectors in $\mathbb{R}^2$. Does there exist a 2×2 matrix which represents the translation of vectors? If so, write it down and justify how you know it represents translations. If not, explain.

6. Show that the product of any two reflection matrices is a rotation matrix. [Hint: You will need the angle subtraction formulas for sine and cosine]. Using this result, give a geometric description of when two reflection matrices will commute.

NAME: **ASSESSMENT PROBLEMS:** GROUPS OF TRANSFORMATIONS (page 1 of 2)

1. Consider the set Δ, given below.

$$\Delta = \left\{ \mathrm{C}(a) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} : a \in \mathbb{R}^+ \right\}$$

   (a) Describe the geometric effect that the matrix $\mathrm{C}(a)$ has on a vector $v \in \mathbb{R}^2$.

   (b) Is Δ an abelian group under matrix multiplication? Demonstrate why or why not.

ASSESSMENT PROBLEMS: GROUPS OF TRANSFORMATIONS (page 2 of 2)

2. Serena is working with the set Σ of rotations about the origin, given below.

$$\Sigma = \left\{ \mathrm{A}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} : \theta \in \mathbb{R} \right\}$$

She knows Σ is a group under multiplication. Reasoning geometrically, Serena argues that the matrices $A(-\frac{\pi}{4})$ and $A(\frac{7\pi}{4})$ both act as inverses of matrix $A(\frac{\pi}{4})$.

(a) What geometric understanding might Serena have of rotations about the origin that enabled her to draw this conclusion without making any computations?

(b) Serena claims that her work above shows that the group of rotations is a counterexample to the claim that all group elements have a unique inverse. Explain the error in Serena's reasoning.

(c) What question would you ask Serena to help her understand her error? Why is your question helpful?