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Inverse Functions and Their Derivatives

Single Variable Calculus

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1.1 Overview and Outline of Lesson

High school students and undergraduates encounter inverse functions in various forms in their mathematics courses. Many students memorize and apply the following technique to find the inverse function of a given function: (1) Write $y = f(x)$; (2) Switch the x and y; and (3) Solve for y. This procedure obscures important relationships and concepts that may help bridge student understandings when these topics are revisited in a more advanced setting. The purpose of this lesson is to further develop undergraduates' conceptual understanding of the relationship between a function and its inverse function and apply this understanding to find derivatives of inverse functions, such as using the derivative of $\tan(x)$ to find the derivative of $\arctan(x)$.

1. Launch—Pre-Activity
   Prior to the lesson, undergraduates complete the Pre-Activity, which focuses on having students think about the relationship between a function and its inverse function. By examining graphs of familiar functions, undergraduates identify when a particular function has an inverse function and what the relationship might be between the domain and range for each function and its inverse function. Instructors can launch the lesson by discussing undergraduates' responses to the Pre-Activity.

2. Explore—Class Activity

   - *Problems 1–2*:
     In Problem 1, undergraduates will compare and contrast three students' methods for finding the inverse of a function. One is the commonly taught method from high school of "switch x and y and solve for y," while the others rely on the mathematical principles underlying the relationship between a function and its inverse function. In Problem 2, undergraduates investigate the limitations of the method "switch x and y and solve for y" in the context of two different situations. They consider whether the other two methods from Problem 1 give rise to the same difficulties.
   - *Problems 3–6*:
     In Problems 3 and 4, the properties of inverse functions and the chain rule are used to compute the derivatives

of $\arcsin(x)$ and $\ln(x)$ in terms of x. Undergraduates reflect on the ways in which the properties of inverse functions were used in these computations to produce the desired form of the derivatives. In Problems 5 and 6, undergraduates write a method for finding the derivative of any inverse function when given a function and its inverse function. They then use this method to write an explicit formula for this derivative and explain why it makes sense.

3. Closure—Wrap-Up
Conclude the lesson by discussing how the properties of inverse functions and the chain rule play an important role in understanding how a function f, its derivative f', and its inverse function f^{-1} can be used to find a formula for the derivative of f^{-1}. Make connections to secondary mathematics by emphasizing how focusing on the properties of inverse functions, in particular the composition of a function and its inverse function (and vice versa), lays the groundwork for establishing the formulas that use these relationships, such as $\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$ for $-1 < x < 1$.

1.2 Alignment with College Curriculum

Inverse functions are encountered in single variable calculus courses in the following ways: (1) defining an inverse function using concepts introduced in precalculus and (2) using inverse functions and inverse relationships to derive the derivatives of inverse functions, in particular finding the derivatives of inverse trigonometric functions. This lesson addresses both of these by (1) reviewing methods to find inverse functions with a focus on the domain and range of the inverse function and the fact that the composition of a function and its inverse function is the identity function; and (2) finding derivatives of inverse functions with a focus on using the chain rule on the composition of a function and its inverse.

1.3 Links to School Mathematics

Finding inverse functions is addressed in high school algebra and precalculus courses, as indicated in the Common Core State Standards for Mathematics (CCSSM, 2010). Frequently, inverse functions are taught as an algebraic manipulation that students memorize, which does not provide natural places to develop automaticity in using composition of functions. This lesson problematizes the common method of "switch x and y and solve for y" to find the inverse of a function by providing sample student work for undergraduates to analyze and discuss.

This lesson highlights:

- Connections between calculus concepts and ideas about functions, including how inverse functions can be leveraged to find certain derivatives;
- The importance of unambiguous mathematical notation when deriving the inverse of a function.

This lesson addresses several mathematical knowledge and practice expectations in common high school standards documents (e.g., CCSSM). To work with functions, high school students are expected to understand the concept of function and use function notation. In particular, most state standards for high school mathematics include detailed expectations related to the definition of function (c.f. CCSS.MATH.CONTENT.HSF.IF.A.1). Additionally, high school students are expected to build functions that model relationships between two quantities. To build appropriate functions, they also learn about function composition and how to find the inverse of a function (c.f. CCSS.MATH.CONTENT.BF.A.1.C, CCSS.MATH.CONTENT.HSF.BF.B.4). This lesson also provides opportunities for prospective teachers to think about the role of the context of a problem, the reasoning of others, and construct sound mathematical arguments.

1.4 Lesson Preparation

Prerequisite Knowledge

Undergraduates should know how to:

- Find the inverse function of a given function algebraically and graphically;
- Use the chain rule.

Learning Objectives

In this lesson, undergraduates will encounter ideas about teaching mathematics, as described in Chapter 1 (see the five types of connections to teaching listed in Table 1.2). In particular, by the end of the lesson undergraduates will be able to:

- Describe the mathematical principles underlying different methods used to compute the inverse of a function;
- Describe the limitations of the method of “switch x and y and solve for y” to find the inverse of a function;
- Compute derivatives of inverse functions using compositions and the chain rule;
- Analyze hypothetical student work and evaluate three different methods used to find the inverse of a function;
- Pose guiding questions to help a hypothetical student understand when certain methods for finding the inverse of a function have mathematical limitations.

Anticipated Length

One 75- to 80-minute class session.

Materials

The following materials are required for this lesson.

- Pre-Activity (assign as homework prior to Class Activity)
- Class Activity
- Computer (for instructor to display a dynamic sketch during the activity)
- Homework Problems (assign at the end of the lesson)
- Assessment Problems (include on quiz or exam after the lesson)

All handouts for this lesson appear at the end of this lesson, and LaTeX files can be downloaded from INSERT URL HERE.

1.5 Instructor Notes and Lesson Annotations

Before the Lesson

Assign the Pre-Activity as homework to be completed in preparation for this lesson. We recommend that you collect this Pre-Activity the day before the lesson so that you can review undergraduates’ responses before you begin the Class Activity. This will help you determine if you need to spend additional time reviewing the solutions to the Pre-Activity with your undergraduates.

Pre-Activity Review (10 minutes)

Discuss undergraduates’ responses to the Pre-Activity, and introduce the lesson by discussing the following connection to teaching. If you wish, provide specific examples of situations in which the existence of an inverse function, in particular, is important.

Discuss This Connection to Teaching

Functions are an important component of school mathematics. A robust understanding of functions provides students with foundational tools for success in calculus and other courses that rely on quantitative thinking and determining relationships between quantities. Despite the importance of building a strong conceptual understanding of functions in grades 8–12, teachers encounter many challenges teaching functions. For example, students “tend to limit the concept of functions to equations or orderly rules” and “frequently overlook many-to-one correspondences or irregular functions that could be very useful in describing or representing real-world phenomena” (Cooney et al., 2010, p. 1). These challenges make it essential for prospective teachers to cultivate their own powerful understanding of functions.

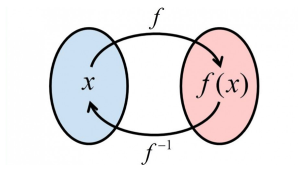
Pre-Activity Problem 1

1. Given a function f, its ***inverse function*** (if it exists) is the function f^{-1} such that $y = f(x)$ if and only if $f^{-1}(y) = x$.

   (a) If we know that $f(♫) = ❤$, what is $f^{-1}(❤)$? What about $f(f^{-1}(❤))$?

   Solution:
   $f^{-1}(❤) = ♫$
   $f(f^{-1}(❤)) = f(♫) = ❤$

   Commentary:
   When discussing 1(a), consider using a function diagram (as seen below) rather than a graph to represent a function and its inverse. Emphasize that the inverse function "undoes" the original function. This behavior is exemplified by their composition.



   (b) If we know a function has an inverse function, what do we know about the properties, behavior, or graph of its inverse function? Create a list of these attributes.

   Sample Responses:
   - It's what you get when you switch x and y and then solve for y. [most common response]
   - All the x and y coordinates are flipped → opposite function.
   - Reflects about $y = x$.
   - The inverse "undoes" f [the original function].
   - Opposite solving steps.
   - It's a function that "reverses" another function.
   - Domain and range flipped.
   - It's whatever function returns the input variable when composed with f [the original function].
   - Function notation: $f^{-1}(x)$.
   - Logarithm and exponential growth graphs are inverses of each other.
   - It's the opposite of f [the original function].
   - It's the reciprocal of f [the original function].

   Commentary:
   For 1(b), ask undergraduates to share their answers. Some common undergraduate responses that are correct, incorrect, or incomplete are provided above. Record their responses on the board or document camera. If the following do not arise in their responses, be sure to add them to the list of attributes since they will be used in discussions that will occur in the rest of the lesson:
   - The graph of the inverse function appears as the reflection over the line $y = x$ of the graph of the original function.
   - An inverse function *"undoes"* the mapping or process induced by its associated function and vice-versa. That is, the composition of a function and its inverse function is the identity, no matter the order of the composition.

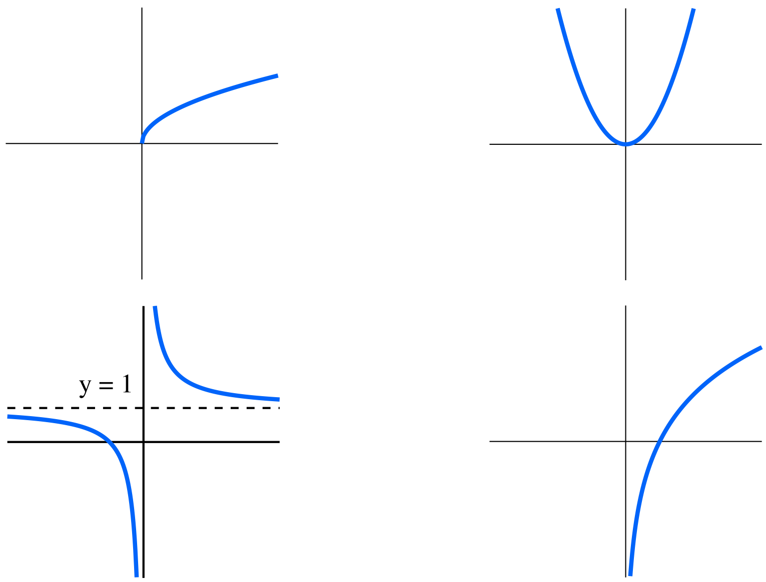
After discussing Problem 1, make sure undergraduates are using correct notation to indicate the inverse of a function and discuss the following connection to teaching, which highlights a common misapplication of the notation used to indicate an inverse function.

Discuss This Connection to Teaching

The fact that $f^{-1}(x) \neq \frac{1}{f(x)}$ runs counter to students' experiences with inverses of real numbers. As a result, students might use their experience with the superscript (-1) in the context of real numbers to incorrectly conclude, for example, that $\sin^{-1}(x) = \frac{1}{\sin(x)}$. Prospective teachers should be able to compare the use of negative exponents when writing 3^{-1} versus $f^{-1}(x)$ and explain why mathematicians have chosen this notation, which in different contexts, yields seemingly "different" actions. See Zazkis & Kontorovich (2016) for further insight.

Pre-Activity Problem 2

2. Consider each of the functions below.



(a) Which of the these functions has an inverse function? Explain how you can know without being given the defining expression.

Solution:
Only the "parabola-looking" function does not have an inverse function. Undergraduate explanations may vary, and include:

- The graph that looks like a parabola is not one-to-one.
- The graph that looks like a parabola does not pass the "horizontal line test."
- If you reflect the graph of the parabola over the line $y = x$, the resulting graph does not represent a function because it fails the vertical line test.

(b) What is the domain and range of each function? What is the domain and range of its inverse function (if it exists)?

Solutions:
Function graph that resembles a square root function:

- $D[f] = (0, \infty)$, $R[f] = (0, \infty)$
- $D[f^{-1}] = (0, \infty)$, $R[f^{-1}] = (0, \infty)$

Function graph that resembles a quadratic function:

- $D[f] = [0, \infty)$, $R[f] = (-\infty, \infty)$
- Does not have an inverse function.

Function graph that resembles a rational function with vertical asymptote at $x = 0$ and horizontal asymptote at $y = 1$:

- $D[f] = (-\infty, 0) \cup (0, \infty)$, $R[f] = (-\infty, 1) \cup (1, \infty)$
- $D[f^{-1}] = (-\infty, 1) \cup (1, \infty)$, $R[f^{-1}] = (-\infty, 0) \cup (0, \infty)$

Function graph that resembles a logarithmic function:

- $D[f] = (0, \infty)$, $R[f] = (-\infty, \infty)$
- $D[f^{-1}] = (-\infty, \infty)$, $R[f^{-1}] = (0, \infty)$

Commentary:
After discussing 2(b), ask the class to compare the domain and range of each function with the domain and range of its inverse function. Do they notice a relationship between the domain and range of a function and the domain and range of its inverse function? Highlight the following relationship and make sure it is added to the list of attributes about functions and their inverse functions on Problem 1b of the Pre-Activity (if it was not there already):

- The domain of f is the range of f^{-1} and the range of f is the domain of f^{-1}.

Wrap up the Pre-Activity by letting undergraduates know that during the Class Activity they will explore how the properties of functions and their inverse functions can be used to determine the derivative of an inverse function by using only the derivative of the original function.

Class Activity: Problems 1–2 (30 minutes)

Distribute the Class Activity and discuss the following connection to teaching.

Discuss This Connection to Teaching

Problem 1 gives undergraduates opportunities to analyze other students' thinking in order to develop their skills in understanding school student thinking. All undergraduates (especially prospective teachers) should examine how others use, reason with, and communicate mathematics. In addition, Problems 1 and 2 of the Class Activity lay the groundwork for problematizing the method of "switch x and y and solve for y" [without attending to the corresponding swap of domain and range] for finding an inverse of a function represented by $y = f(x)$. Raising awareness regarding how this method may not advance understanding or connect to conceptual ideas in calculus for prospective teachers, in particular, may influence their own teaching of inverse functions in the future.

Give undergraduates a few minutes to study Problem 1 before directing them to work in groups on this problem. (See Chapter 1 for guidance on facilitating group work and selecting and sequencing student work for use in whole-class discussion.) In this problem, undergraduates will examine the work of three hypothetical students, Alex, Jordan, and Kelly, who used different methods to find the inverse of a function. If undergraduates focus on whether the algebra in each student's work is correct, let them know that the computations are correct and that they should focus on the methods each student is using to find the inverse function.

Alex's method in Problem 1 is not always appropriate, and even when it is appropriate mathematically, its use may obscure the meaning behind the variables and can lead to confusion in contextualized problems. Undergraduates will see specific examples of this shortcoming in Problem 2. Jordan's and Kelly's method makes use of the definition of inverse functions. By recognizing that the inverse function of $y = f(x)$ is $x = f^{-1}(y)$, learners can make sense of inverse functions in multiple mathematical contexts including real world data analysis and modeling (Wilson et al., 2016). By discussing the relationship between different input values and output values of a function and its inverse function, undergraduates can focus on the concept that inverse functions undo the mapping produced by the original function. In addition, analyzing three different (although incomplete) sample student approaches to finding the inverse function puts undergraduates in a teaching situation by having them analyze *student work. This analysis may sharpen conceptual understanding of inverse functions (e.g., understanding why and how different methods work and how they are grounded in mathematical operations) as well as develop skills for examining student work for mathematical understanding (Cooney et al., 2010).*

Class Activity Problem 1

1. Consider how Alex, Jordan, and Kelly found the inverse function of $f(x) = \frac{2}{3}x + 1$.

Alex's Work	Jordan's Work	Kelly's Work
$y = 2/3\,x + 1$ $x = 2/3\,y + 1$ $x - 1 = 2/3\,y$ $\frac{x-1}{2/3} = y$ $\boxed{\frac{3}{2}(x-1) = y}$	$f \circ f^{-1}(y) = y$ $f(x) = \frac{2}{3}x + 1$ So: $f(f^{-1}(y)) = \frac{2}{3}f^{-1}(y) + 1 = y$ $\frac{2}{3}f^{-1}(y) = y - 1$ $\frac{3}{2}\cdot\frac{2}{3}f^{-1}(y) = \frac{3}{2}(y-1)$ $\boxed{f^{-1}(y) = \frac{3}{2}(y-1)}$	$y = \frac{2x}{3} + 1$ $y - 1 = \frac{2x}{3}$ $3(y-1) = 2x$ $\frac{3(y-1)}{2} = x$ But $f(x) = y \Rightarrow$ $f^{-1}(f(x)) = f^{-1}(y) \Rightarrow$ $x = f^{-1}(y)$ So $\boxed{f^{-1}(y) = \frac{3(y-1)}{2}}$

Compare and contrast the key mathematical ideas used by Alex, Jordan, and Kelly to find the inverse function of $f(x) = \frac{2}{3}x + 1$. Make sure to identify which properties of inverse functions each student uses, if any.

Sample Responses:

- Alex switches x and y and solves for y.
- Jordan uses the fact that the composition of a function and its inverse function is the identity to write an equation involving $f^{-1}(y)$, then solves for that as if it were a variable.
- Kelly solves for x, then uses that fact that $f^{-1}(f(x)) = f^{-1}(y) = x$ to make a substitution.
- Both Jordan and Kelly end up with a function of y, while Alex's method produces a function of x.
- Both Jordan and Kelly use the definition of an inverse function.

Commentary:

From our experience, undergraduates have expressed their familiarity with using Alex's method (i.e., "switch x and y and solve for y"), and some have argued that Jordan or Kelly are "wrong" in some way. An important feature of this problem is to have undergraduates engage in providing a mathematical explanation for *why* they feel this way. This affords them an opportunity to enhance their capacity to make sense of and justify unfamiliar methods. If undergraduates are unfamiliar with Jordan's and Kelly's method, the following questions may help them to make sense of Jordan's and Kelly's work.

- How is Jordan/Kelly starting the problem?
- Why do you think Jordan/Kelly is using $f^{-1}(y)$? What property of inverse functions does this highlight?
- Which methods make use of the definition of the inverse function that we saw in the Pre-Activity?
- What understanding of inverse functions does Jordan/Kelly have that Alex may not?
- Why does Jordan's/Kelly's/Alex's method work?
- Why do you think Jordan's/Kelly's/Alex's method does not work?

After undergraduates have had some time to work through Problem 1, gather the class back together for a discussion. First ask undergraduates to report out their responses. As you facilitate the discussion, call attention to explanations that involve the properties of inverse functions that they noted while working to answer the problem. Consider compiling a master list of these "mathematical ideas" as undergraduates volunteer them. Undergraduates will refer back to these ideas during Problem 4 of the Class Activity and being able to quickly remember the highlights of the discussion will help streamline the process.

Consider asking the following questions, which can prompt discussion about the applicability of each method:

- Which method would you most likely use? Why?
- Which method makes the most sense to you? Why?
- Is Alex's solution "the same" as the other two? (i.e., Does this difference "matter"?) Why/Why not?

Instruct undergraduates to work in groups on Problems 2(a) and 2(b) and emphasize that the computations in the students' work are correct which redirects students' attention to the conceptual ideas instead of their focusing on the computations. From our experience, we have found that undergraduates did not immediately know the best way to answer these questions. Allow sufficient time to work in groups so that undergraduates have time to recoup if they initially misinterpreted the problem.

Class Activity Problem 2 : Parts a & b

2. Now consider two problems where a high school student used Alex's method of switching the variables and solving for the dependent variable to find the inverse function.

Find the inverse function of $T(C) = \frac{9}{5}C + 32$ where C is the temperature in Celsius and $F = T(C)$ gives the temperature in Fahrenheit.	Find the inverse function of $f(x) = \frac{2x+1}{x-1}$ for $x \neq 1$.
Let $F = \frac{9}{5}C + 32$ $C = \frac{9}{5}F + 32$ $C - 32 = \frac{9}{5}F$ $\frac{5}{9}(C-32) = F$	$y = \frac{2x+1}{x-1}, \quad x \neq 1$ $x = \frac{2y+1}{y-1}$ $(y-1)x = 2y+1$ $yx - x = 2y+1$ $yx - 2y = 1 + x$ $y(x-2) = x+1$ $y = \frac{x+1}{x-2}, \quad x \neq 1$

(a) Describe why the student's work for the temperature function is problematic.

Sample Responses:

- No inverse notation is used.
- We should not switch F and C because they represent different units.
- We wanted a function where we plug in Fahrenheit and get the temperature in Celsius; this would "undo" the original function.
- We appear to have two different functions for changing Celsius to Fahrenheit which behave differently.

Commentary:

Below are some questions that you might ask undergraduates to facilitate their group conversations:

- What does the inverse function mean in this context?
- Is the property that an inverse function "undoes" the original function preserved? How can we check?
- What are the units attached to F and C? Are they preserved when we switch F and C?
- How might we fix this work so that these issues are resolved?

(b) Describe why the student's work for the rational function is problematic.

Sample Responses:

- $x \neq 1$ is an appropriate restriction on the first function, but does not make sense for its inverse function.
- The domain and the range of the function and its inverse were not "switched"—the last line has an unnecessary restriction on x, but without the necessary restriction of $x \neq 2$.

Commentary:
Below are some questions that you might ask undergraduates to facilitate their group conversations on Problem 2(b):

- Where are the horizontal and vertical asymptotes of the original function? Of the inverse function?
- Look back at the rational looking function from Problem 2 of the Pre-Activity. How did its domain and range relate to the domain and range of its inverse function?
- How might we fix this work so that these issues are resolved?

Before moving on to Problem 2(c), call the class back together and allow groups to share and discuss their answers to Problems 2(a) and 2(b).

Class Activity Problem 2 : Part c

(c) What are the limitations of using Alex's method of switching the variables and solving for the dependent variable to find an inverse function? Would Jordan and Kelly have the same problem(s)? Explain.

Sample Responses:

- Variables might have an associated unit which we ignore when we switch them.
- Switching the variables is misleading if we also do not account for the change in domain/range when considering what x and y now represent in the context of the inverse function.
- Jordan and Kelly would not have the same problem because their notation would preserve the identity of the variables and would switch the domain and range.
- When using Jordan's method, we use the fact that the composition of a function and its inverse function is the identity, which implicitly requires that the range of the inner function be the domain of the outer function. Jordan's method also does not switch the units attached to each variable.
- When using Kelly's method, the substitution $x = f^{-1}(y)$ reminds us that the inverse function "undoes" the original function by using the range of f as its domain. Kelly's method also does not mix up the units which may be attached to each variable.

Commentary:
Note that there is some overlap between Problem 2(c) and Problems 2(a) and 2(b). The following prompts can be used to facilitate a class discussion.

- "Is the y at the end the same as the y at the beginning? Why or why not?"
- For many functions, the meaning of the variables are important. Switching them confuses their intended meaning.
- Solving by switching x and y hides these properties of inverse functions:
  - The inverse function *undoes* whatever the function does. If the original function maps x values to y values, the inverse function should send y values back to x values.
  - The domain of the function is the range of its inverse function and vice versa.
- Jordan and Kelly avoid these problems by indicating that the inverse function is a function of y, not x.

After discussing Problem 2(c), emphasize to your class that Alex's method is not, strictly speaking, incorrect; but that it does come with certain drawbacks. Discuss the following connection to teaching to highlight this point.

Discuss This Connection to Teaching

Finding the inverse of a function is taught in second year school algebra (i.e., Algebra II in most states in the United States), and it is often taught with a procedural emphasis. Students memorize a procedure (e.g., "switch x and y and solve for y") and apply it to find the inverse of a given function. Although this procedure "works" when students are working with linear functions, students tend to struggle when working with other functions, such as transcendental functions (Teuscher et al., 2018). In addition, procedural approaches often are not grounded in the mathematical operations associated with the relationship between a function and its inverse function, may hinder a learner's understanding of the derivatives of inverse functions as they relate to the original function, and may inadvertently limit students' contextual practice in composition of functions (Wilson et al., 2016).

Class Activity: Problems 3–6 (30 minutes)

Introduce the next part of the Class Activity by explaining to undergraduates that they will use the previous explorations on inverse functions in order to learn to find derivatives of inverse functions. Explain that to find such a derivative, they will need to already know the derivative of the original function and will use function compositions and the relationship between a function and its inverse function.

First, tell your class to only work on Problem 3(a) in groups. Before undergraduates begin, ask them what they know about the inverse function of $y = \arcsin(x)$.

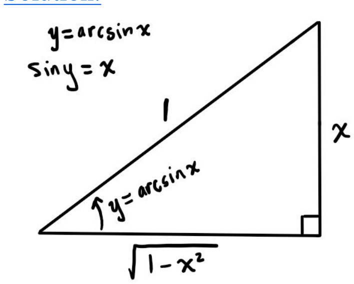
- Ask undergraduates what x and y mean in the context of the problem. You may need to remind them that $\arcsin(x) = \sin^{-1}(x)$ and that $\arcsin(x)$ is the angle whose sine value is x.
- It may be helpful to refer back to the definition of an inverse function to prompt undergraduates to conclude that $\sin(y) = x$.

Class Activity Problem 3 : Part a

3. We can use the properties of inverse functions to find their derivatives with respect to x.

   (a) Draw a right triangle that illustrates the relationship $y = \arcsin(x)$ for $0 < x < 1$.

   Solution:



   Commentary:
   From our experience, an undergraduate who uses Jordan's method will likely feel more comfortable working with $\sin(\arcsin(x)) = x$ or other mathematical statements involving the composition of functions. Jordan's method also requires the manipulation of function notation as a mathematical object, a skill which many undergraduates struggle to develop.

After most groups have a reasonable sketch of the triangle (such as the one presented in the solution above), share an undergraduate's correct drawing and discuss the following connection to teaching. You may wish to emphasize that this triangle model only considers values of y within the interval $(0, \frac{\pi}{2})$, but can be extended to account for $y \in (-\frac{\pi}{2}, 0]$. Alternatively, consider asking your class to consider why we might need to make this restriction.

Discuss This Connection to Teaching

High school students typically create these types of right triangles to illustrate how the values of trigonometric functions of an acute angle are ratios of the lengths of the sides of a right triangle. These images can be used to help students compute all six trigonometric functions of a given angle and examine other characteristics of trigonometric functions.

Now allow your class to complete Problem 3(b) by finding the derivative of $y = \arcsin(x)$ in terms of x. You may want to hint that they should take the derivative of both sides of the given equation. Remind them to use the picture to make substitutions when necessary to write their final answer as a function of x.

Class Activity Problem 3 : Part b

(b) Use the fact that $\sin(\arcsin(x)) = x$ and the chain rule to compute $\frac{d}{dx}\arcsin(x)$ in terms of x.

Solution:

$$\begin{aligned} \sin(\arcsin(x)) &= x \\ \frac{d}{dx}\sin(\arcsin(x)) &= \frac{d}{dx}x \\ \cos(\arcsin(x)) \cdot \frac{d}{dx}\arcsin(x) &= 1 \\ \frac{d}{dx}\arcsin(x) &= \frac{1}{\cos(\arcsin(x))} \\ \frac{d}{dx}\arcsin(x) &= \frac{1}{\cos(y)} \\ \frac{d}{dx}\arcsin(x) &= \frac{1}{\sqrt{1-x^2}} \end{aligned}$$

Commentary:
The following prompts can be used to facilitate a class discussion.

- How does the chain rule come into play in this problem?
- How did you use your picture while you worked? In what ways was your picture helpful as you attempt to write the derivative of $\arcsin(x)$ in terms of x (that is, "by eliminating or not involving a trigonometric function")?

Instruct undergraduates to work on Problem 3(c). Encourage them to use their work on Problem 3(b) as a guide, telling them that the way the chain rule is involved in each part is largely the same.

Class Activity Problem 3 : Part c

(c) Use the fact that $e^{\ln(x)} = x$ and the chain rule to show that $\frac{d}{dx}\ln(x) = \frac{1}{x}$.

Solution:

$$\begin{aligned} \frac{d}{dx}e^{\ln(x)} &= \frac{d}{dx}x \\ e^{\ln(x)} \cdot \frac{d}{dx}\ln(x) &= 1 \\ \frac{d}{dx}\ln(x) &= \frac{1}{e^{\ln(x)}} \\ \frac{d}{dx}\ln(x) &= \frac{1}{x} \end{aligned}$$

Commentary:
You may or may not want to re-engage the whole class for a discussion depending on the proficiency they demonstrate as they solve Problem 3(c) in their smaller groups. Some discussion questions you might use include:

- The composition of two functions in Problem 3(c) is slightly less obvious. Which is the interior function and which is the exterior?
- For what values of x is this formula for the derivative valid? Why?

Transition to the final three problems in the Class Activity by appealing to the need for a generalized formula; that is, while it is nice to be able to find the derivatives for particular functions, as in Problem 3, it would certainly be useful if we could find a "nice" relationship between the derivative of any function to the derivative of its inverse.

Class Activity Problem 4

4. What key mathematical idea(s) from Problem 1 did we use to find the derivatives in Problem 3?

Solution:
We used the idea from Jordan and Kelly that a function composed with its inverse produces the identity.

Commentary:
If you chose to write a master list of "mathematical ideas" on the board during Problem 1, you might choose not to break out into smaller groups before the class discussion on this problem. No matter how the discussion begins, make sure that undergraduates recognize that in both derivations from Problem 3, they had to begin with the idea that the composition of a function with its inverse function is the identity. This allowed them to apply chain rule and solve for the required derivative.

Instruct undergraduates to work on Problem 5(a) and 5(b) and verify that they are using chain rule correctly. In Problem 5(a), undergraduates need to write a few sentences *describing* the process they would use to compute $\frac{d}{dx}g(x)$ for any function f with inverse function g, and in Problem 5(b), they will apply that process.

Class Activity Problem 5 : Parts a & b

5. Let f be a function with inverse function g.

(a) In the form of written sentences, describe how you would use the fact that $f(g(x)) = x$ and the chain rule to compute $\frac{d}{dx}g(x)$ for any function f with inverse function g.

Sample Response:
You would take the derivative of both sides of $f(g(x)) = x$ with respect to x, using the chain rule on the left hand side. The derivative of both sides of the equation, $f(g(x)) = x$ is $f'(g(x))\cdot g'(x) = 1$. We are then able to divide both sides of the equation by $f'(g(x))$, provided it is not equal to zero, in order to arrive at an expression for the derivative of a general inverse function.

(b) Use the procedure you wrote above to compute $\frac{d}{dx}g(x)$ for any function f with inverse function g.

Solution:

$$\begin{aligned} f(g(x)) &= x \\ \frac{d}{dx}[f(g(x))] &= \frac{d}{dx}x \\ f'(g(x))\cdot g'(x) &= 1 \\ g'(x) &= \frac{1}{f'(g(x))} \text{ where } f'(g(x)) \neq 0 \end{aligned}$$

Ask undergraduates to complete Problem 5(c), and have this dynamic sketch (`https://www.desmos.com/calculator/6fegvrt5xv`) in a place where it is visible to the whole class in order to help stimulate their discussion. When groups begin working on Problem 5(c), listen to the group discussions and find an appropriate time to bring the whole class together to demonstrate the features of the dynamic sketch. At this point, discuss the connection to teaching below.

Discuss This Connection to Teaching

Using multiple representations of functions (in this case, an algebraic representation and a graph) may help students connect mathematical ideas and strengthen understanding. Emphasizing qualitative ways to make sense of algebraic and graphical representations provides students with more experiences making sense of function behavior, similar to the sense-making needed for understanding rates of change and accumulation in calculus.

Class Activity Problem 5 : Part c

(c) Why does the result in Problem 5(b) make sense given what we know about visualizing derivatives and the graphs of inverse functions?

Sample Response:
The derivative at a point represents the slope of the tangent line at that point. If f has an inverse function, the graph of f^{-1} is the graph of f reflected over the line $y = x$. Finally, we can also easily see that the reflection of a linear function reflected over the line $y = x$ has the reciprocal slope of the original line. Putting all these facts together, it makes sense that the derivative of the inverse function is some kind of reciprocal, since it is the slope of the reflection of a tangent line.

Commentary:
Consider posing the following questions as undergraduates continue their work in groups:

- How might we visualize the derivative of a function on its graph?
- How does the graph of a function relate to the graph of its inverse function?
- Given an arbitrary linear function, $f(x) = mx + b$, what is its inverse function? What is the slope of this inverse function? How do the slope of the function and the slope of its inverse function relate?

Finally, for Problem 6, allow groups only a few minutes to try their procedure from Problems 5(a) and 5(b) on the composition $g(f(x))$ to see if it is still possible to solve for $g'(x)$. Facilitate a short discussion in which undergraduates establish that the function we "want the derivative of" should be the "inner" or "inside" function. Compare this to our method of solving for the derivative of arcsine or natural logarithm.

Class Activity Problem 6

6. What happens when we reverse the order of the composition in the previous problem before we differentiate? Can we still find a formula for $\frac{d}{dx}g(x)$? Why or why not?

Solution:
Taking the derivative of both sides of the equation, $g(f(x)) = x$, with respect to x gives us $g'(f(x)) \cdot f'(x) = 1$. This doesn't give us an easy way to solve for the derivative of g since the chain rule "pulled out" the derivative of f instead.

Wrap-Up (5 minutes)

Conclude the lesson by revisiting some of the ideas that were discussed during the lesson, emphasizing how they connect undergraduate calculus to secondary mathematics concepts. Some points to consider include:

- Inverse functions are more than just the result of an algorithm. They have important properties, such as:
  - The composition of a function with its inverse function is the identity and vice-versa.
  - The domain of the original function is the range of the inverse function and vice-versa.
  - The inverse function "undoes" the original function (and vice-versa!).
- In particular, we can use these ideas as a tool to help us find formulas for derivatives of functions, such as the inverse trigonometric functions, for which direct use of the definition of the derivative yields expressions that may require tools beyond first-semester calculus to resolve.
- The method of "switch x and y and solve for y" comes with certain drawbacks:
  - It disregards units which may be attached to the original variables.
  - It glosses over the fact that the domain and range must be appropriately adjusted.
  - It does not provide an opportunity for students to revisit composition of functions in this setting as well as develop computational fluency with this operation.

  Prospective teachers who understand the importance of the role of function composition in working fluently with inverse functions in later courses are better equipped to convey this understanding to their students, who in turn benefit from increased awareness of the power of inverse functions.

At the end of the lesson, you can collect exit tickets if you choose. See Chapter 1 for guidance on using exit tickets in instruction.

Homework Problems

At the end of the lesson, assign the following homework problems.

In Problem 1, undergraduates must attend to the fact that the domain and range of the original function are the range and domain of the inverse function, respectively. Asking leading questions to a hypothetical student requires undergraduates to identify the underlying contradictions that arise from failing to recognize this fact.

Homework Problem 1

1. A theater concludes that their total revenue for the week is a function of the number of tickets they sell. They use the equation $R(t) = 15t - 100$ to represent this relationship, where t is the number of tickets sold. Akira uses the method of "switch x and y and solve for y" to find the inverse function of the relationship described above.

   Let $r = 15t - 100$

   $t = 15r - 100$

   $t + 100 = 15r$

   $r = \frac{1}{15}(t + 100)$

   What two questions would you ask Akira to help them see the limitations of their work? Why would your questions be helpful?

   Sample Response:

   - If you are asked to calculate the revenue from 20 tickets sold, which equation would you use?
   - The equation $r = \frac{1}{15}(t + 100)$ implies that for $t = 0, r > 0$. Does this make sense in the context of the situation?

My two questions are helpful because they would show Akira that the notation used in their "solution" is misleading. By switching the variable names, they are no longer representing the situation as intended.

Solving Problem 2 requires an understanding of the fact that inverse functions "undo" the process of the original function. It also helps address the possible misconception that $(f \circ g)^{-1} = f^{-1} \circ g^{-1}$.

Homework Problem 2

2. Using the properties of inverse functions, find the inverse function of the composite function $h(x) = f(g(x))$, where both f and g are known to have inverse functions.

Solution:

$$\begin{aligned} h(x) &= f(g(x)) \\ f^{-1}(h(x)) &= f^{-1}(f(g(x))) \\ f^{-1}(h(x)) &= g(x) \\ g^{-1}(f^{-1}(h(x))) &= g^{-1}(g(x)) \\ g^{-1}(f^{-1}(h(x))) &= x \end{aligned}$$

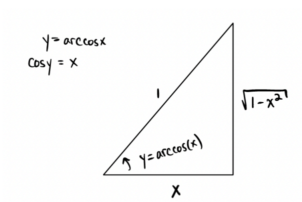
So, if $j(x) = g^{-1}(f^{-1}(x))$, then $j(h(x)) = x$. We can also show that $h(j(x)) = x$. By definition, we then have that $h^{-1}(x) = j(x) = g^{-1}(f^{-1}(x))$

Problem 3 gives undergraduates the opportunity to practice the techniques introduced in the lesson and look for patterns in their results. In particular, they will see how the derivatives of inverse trigonometric functions relate to the derivatives of their inverse cofunctions.

Homework Problem 3

3. Find a formula for $\frac{d}{dx}\arccos(x)$ in terms of x. Compare this formula to $\frac{d}{dx}\arcsin(x)$ (from Problem 3(b) in the Class Activity). What do you notice about the two derivatives?

Solution:



$$\begin{aligned} \cos(\arccos(x)) &= x \\ \frac{d}{dx}\cos(\arccos(x)) &= \frac{d}{dx}x \\ -\sin(\arccos(x)) \cdot \frac{d}{dx}\arccos(x) &= 1 \\ \frac{d}{dx}\arccos(x) &= \frac{-1}{\sin(\arccos(x))} \\ \frac{d}{dx}\arccos(x) &= \frac{-1}{\sin(y)} \\ \frac{d}{dx}\arccos(x) &= \frac{-1}{\sqrt{1-x^2}} \end{aligned}$$

The derivatives only differ by their sign; that is, $\frac{d}{dx}\arccos(x) = -\left(\frac{d}{dx}\arcsin(x)\right)$.

Problem 4 offers an interesting extension that examines functions with fixed points. Graphical representations of inverse functions, using equal-sized scales on each axis, exhibit symmetry about the line $y = x$. Fixed points also involve this identity line, and comparing the two concepts in Problem 4 helps undergraduates develop well-connected representations of both ideas.

Homework Problem 4

4. We call a point a in the domain of a function f a *fixed point* if $f(a) = a$.

   (a) Give an example of a continuous function with no fixed points.

   Sample Responses:
   - $f(x) = \ln(x)$
   - Any line parallel to $y = x$, such as $f(x) = x + 1$.
   - Any parabola which has been shifted up so that its graph no longer intersect the line $y = x$, such as $f(x) = x^2 + 100$
   - Any function f for which its graph lies strictly above the line $y = x$ or strictly below the line $y = x$.

   (b) Give an example of a continuous function with precisely one fixed point.

   Sample Responses:
   - $f(x) = \ln(x) + 1$ has exactly 1 fixed point at $(1, 1)$.
   - Any line that is not parallel to $y = x$, such as $f(x) = 2x$, will have one fixed point.
   - $f(x) = \frac{1}{2}(x + \frac{1}{2})^2$ has a fixed point at $\left(\frac{1}{2}, \frac{1}{2}\right)$.

   (c) For some continuous function f, assume its inverse function exists and has a fixed point (i.e., $f^{-1}(b) = b$). Does f have a fixed point? If so, for what value of x?

   Solution:
   If $f^{-1}(b) = b$, then $b = f(b)$ by the definition of an inverse function. This means any fixed point for the inverse function is also a fixed point for the original. Graphing the function and its inverse makes this clear, since they are reflections of each other over the line $y = x$.

Assessment Problems

The following two problems address ideas explored in the lesson, with a focus on connections to teaching and mathematical content. You can include these problems as part of your usual course quizzes or exams.

Assessment Problems 1 & 2

1. A classmate is finding the formula of an inverse function and does not yet understand why the method of "switch x and y and solve for y" is problematic. Provide an example of a situation in which this approach is problematic and explain why.

   Sample Responses:
   - An equation like $y = x^2 + 2^x$ that mixes "types" of functions (e.g., quadratic and exponential) defy our solution strategies. We can neither "isolate the x^2 and take square roots" nor "isolate the 2^x and use logarithms." So we can find examples where the expression in terms of x makes it difficult or impossible to solve for y after a variable switch.
   - In the case of a function that relates two real-world quantities together, operations that ignore the units associated with the variables often leads to inconsistencies and incorrect conclusions (for example, the linear relationship between Fahrenheit and Celsius as seen in Problem 2(a) from the Class Activity). Switching the variables also interchanges the units, which could lead to confusion.

- The domain and the range of a function and its inverse are switched. If we switch x and y and solve for y, this obscures the fact that the x and y are "different" in a fundamental way than the variables we began with. For example, $y = \frac{1}{x+1}$ has a vertical asymptote at $x = -1$; if we write its inverse function (after "switching") as $y = \frac{1}{x} - 1$, we might also think $x \neq 0$, which isn't true in the case of the original function.

2. Show the steps used to find a formula for $\frac{d}{dx}\arctan(x)$ in terms of x. Express $\frac{d}{dx}\arctan(x)$ as a rational function.

   Solution:
   This is finding a formula for $\left(f^{-1}\right)'(x)$ where $f^{-1}(x) = \arctan(x)$ (and $f(x) = \tan(x)$). We also know that $\tan(\arctan(x)) = x$. We then take the derivative of both sides of this equation.

$$\begin{aligned} \tan(\arctan(x)) &= x \\ \frac{d}{dx}\tan(\arctan(x)) &= \frac{d}{dx}x \\ \sec^2(\arctan(x))\left(\frac{d}{dx}\arctan(x)\right) &= 1 \\ \frac{d}{dx}\arctan(x) &= \frac{1}{\sec^2(\arctan(x))} \end{aligned}$$

   The trig identity $\sec^2(\theta) = 1+\tan^2(\theta)$ allows us to rewrite the above as $\frac{d}{dx}\arctan(x) = \frac{1}{1+\tan^2(\arctan(x))}$. We use $\tan(\arctan(x)) = x$ to rewrite this as $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$.

1.6 References

[1] Cooney, T. J., Beckmann, S., & Lloyd, G. M. (2010). *Developing essential understanding of functions for teaching mathematics in grades 9–12*. National Council of Teachers of Mathematics.

[2] National Governors Association Center for Best Practices & Council of Chief State School Officers (2010). *Common Core State Standards for Mathematics.* Authors. Retrieved from `http://www.corestandards.org/`

[3] Teuscher, D., Palsky, K., & Palfreyman, C. Y. (2018). Inverse functions: Why switch the variable? *The Mathematics Teacher*, *111*(5), 374–381.

[4] Wilson, F., Adamson, S., Cox, T., & O'Bryan, A. (2016, November 28). *Inverse functions: We're teaching it all wrong!*. AMS Blogs on Teaching & Learning Math. Retrieved from `https://blogs.ams.org/matheducation/2016/11/28/inverse-functions-were-teaching-it-all-wrong/`

[5] Zazkis, R. & Kontorovich, I. (2016). A curious case of superscript (-1): Prospective secondary mathematics teachers explain. *The Journal of Mathematical Behavior*, *43*, 98–110.

1.7 Lesson Handouts

Handouts for use during instruction are included on the pages that follow. LaTeX files for these handouts can be downloaded from INSERT URL HERE.

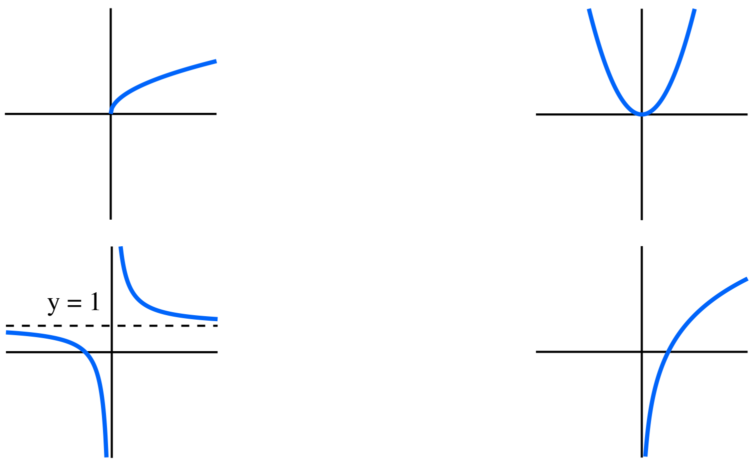
NAME: **PRE-ACTIVITY:** INVERSE FUNCTIONS AND THEIR DERIVATIVES (page 1 of 1)

1. Given a function f, its ***inverse function*** (if it exists) is the function f^{-1} such that $y = f(x)$ if and only if $f^{-1}(y) = x$.

   (a) If we know that $f($♫$) =$ ❤, what is $f^{-1}($❤$)$? What about $f(f^{-1}($❤$))$?

   (b) If we know a function has an inverse function, what do we know about the properties, behavior, or graph of its inverse function? Create a list of these attributes.

2. Consider each of the functions below.



   (a) Which of the these functions has an inverse function? Explain how you can know without being given the defining expression.

   (b) What is the domain and range of each function? What is the domain and range of its inverse function (if it exists)?

NAME: **CLASS ACTIVITY:** INVERSE FUNCTIONS AND THEIR DERIVATIVES (page 1 of 4)

1. Consider how Alex, Jordan, and Kelly found the inverse function of $f(x) = \frac{2}{3}x + 1$.

Alex's Work	Jordan's Work	Kelly's Work
$y = 2/3\, x + 1$ $x = 2/3\, y + 1$ $x - 1 = 2/3\, y$ $\frac{x-1}{2/3} = y$ $\boxed{\frac{3}{2}(x-1) = y}$	$f \circ f^{-1}(y) = y$ $f(x) = \frac{2}{3}x + 1$ So: $f(f^{-1}(y)) = \frac{2}{3}f^{-1}(y) + 1 = y$ $\frac{2}{3}f^{-1}(y) = y - 1$ $\frac{3}{2} \cdot \frac{2}{3} f^{-1}(y) = \frac{3}{2}(y-1)$ $\boxed{f^{-1}(y) = \frac{3}{2}(y-1)}$	$y = \frac{2x}{3} + 1$ $y - 1 = \frac{2x}{3}$ $3(y-1) = 2x$ $\frac{3(y-1)}{2} = x$ But $f(x) = y \Rightarrow$ $f^{-1}(f(x)) = f^{-1}(y) \Rightarrow$ $x = f^{-1}(y)$ So $\boxed{f^{-1}(y) = \frac{3(y-1)}{2}}$

Compare and contrast the key mathematical ideas used by Alex, Jordan, and Kelly to find the inverse function of $f(x) = \frac{2}{3}x + 1$. Make sure to identify which properties of inverse functions each student uses, if any.

2. Now consider two problems where a high school student used Alex's method of switching the variables and solving for the dependent variable to find the inverse function.

Find the inverse function of $T(C) = \frac{9}{5}C + 32$ where C is the temperature in Celsius and $F = T(C)$ gives the temperature in Fahrenheit.	Find the inverse function of $f(x) = \frac{2x+1}{x-1}$ for $x \neq 1$.
Let $F = \frac{9}{5}C + 32$ $C = \frac{9}{5}F + 32$ $C - 32 = \frac{9}{5}F$ $\frac{5}{9}(C-32) = F$	$y = \frac{2x+1}{x-1}$, $x \neq 1$ $x = \frac{2y+1}{y-1}$ $(y-1)x = 2y+1$ $yx - x = 2y + 1$ $yx - 2y = 1 + x$ $y(x-2) = x+1$ $y = \frac{x+1}{x-2}$, $x \neq 1$

(a) Describe why the student's work for the temperature function is problematic.

(b) Describe why the student's work for the rational function is problematic.

(c) What are the limitations of using Alex's method of switching the variables and solving for the dependent variable to find an inverse function? Would Jordan and Kelly have the same problem(s)? Explain.

3. We can use the properties of inverse functions to find their derivatives with respect to x.

   (a) Draw a right triangle that illustrates the relationship $y = \arcsin(x)$ for $0 < x < 1$.

   (b) Use the fact that $\sin(\arcsin(x)) = x$ and the chain rule to compute $\frac{d}{dx}\arcsin(x)$ in terms of x.

   (c) Use the fact that $e^{\ln(x)} = x$ and the chain rule to show that $\frac{d}{dx}.\ln(x) = \frac{1}{x}$

4. What key mathematical idea(s) from Problem 1 did we use to find the derivatives in Problem 3?

5. Let f be a function with inverse function g.

   (a) In the form of written sentences, describe how you would use the fact that $f(g(x)) = x$ and the chain rule to compute $\frac{d}{dx}g(x)$ for any function f with inverse function g.

   (b) Use the procedure you wrote above to compute $\frac{d}{dx}g(x)$ for any function f with inverse function g.

   (c) Why does the result in Problem 5(b) make sense given what we know about visualizing derivatives and the graphs of inverse functions?

6. What happens when we reverse the order of the composition in the previous problem before we differentiate? Can we still find a formula for $\frac{d}{dx}g(x)$? Why or why not?

NAME: **HOMEWORK PROBLEMS:** INVERSE FUNCTIONS AND THEIR DERIVATIVES (page 1 of 1)

1. A theater concludes that their total revenue for the week is a function of the number of tickets they sell. They use the equation $R(t) = 15t - 100$ to represent this relationship, where t is the number of tickets sold. Akira uses the method of "switch x and y and solve for y" to find the inverse function of the relationship described above.

   Let $r = 15t - 100$

   $t = 15r - 100$

   $t + 100 = 15r$

   $r = \frac{1}{15}(t + 100)$

   What two questions would you ask Akira to help them see the limitations of their work? Why would your questions be helpful?

2. Using the properties of inverse functions, find the inverse function of the composite function $h(x) = f(g(x))$, where both f and g are known to have inverse functions.

3. Find a formula for $\frac{d}{dx}\arccos(x)$ in terms of x. Compare this formula to $\frac{d}{dx}\arcsin(x)$ (from Problem 3(b) in the Class Activity). What do you notice about the two derivatives?

4. We call a point a in the domain of a function f a *fixed point* if $f(a) = a$.
   (a) Give an example of a continuous function with no fixed points.
   (b) Give an example of a continuous function with precisely one fixed point.
   (c) For some continuous function f, assume its inverse function exists and has a fixed point (i.e., $f^{-1}(b) = b$). Does f have a fixed point? If so, for what value of x?

NAME: **ASSESSMENT PROBLEMS:** INVERSE FUNCTIONS AND THEIR DERIVATIVES (page 1 of 1)

1. A classmate is finding the formula of an inverse function and does not yet understand why the method of "switch x and y and solve for y" is problematic. Provide an example of a situation in which this approach is problematic and explain why.

2. Show the steps used to find a formula for $\frac{d}{dx}\arctan(x)$ in terms of x. Express $\frac{d}{dx}\arctan(x)$ as a rational function.