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Logarithms and Isomorphisms

Abstract (Modern) Algebra I

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1.1 Overview and Outline of Lesson

The defining characteristic of the logarithm, as taught in high school, is that it is the inverse of the exponential function and it is used as a tool for solving exponential equations. Often overlooked is that the logarithm was created as an isomorphism between the set of positive real numbers under multiplication and the set of all real numbers under addition. In this lesson, undergraduates learn that the logarithm was originally devised as a computational aid and that its use in this manner is possible because it is both a group homomorphism and a group isomorphism. This lesson highlights this distinction by contrasting the logarithm with the determinant function on invertible matrices, which fails to act as a computational aid because it is a group homomorphism but not a group isomorphism. In this lesson, undergraduates also explore the structure-preserving nature of isomorphisms.

1. Launch—Pre-Activity:
   Undergraduates complete this assignment prior to the lesson. The Pre-Activity addresses the historical genesis of the logarithm and other familiar functions that are not homomorphisms. Undergraduates investigate how to use a logarithm table to approximate different values of the function $\log(x)$.

2. Explore—Class Activity:

   - *Problems 1–4*
     After using a larger logarithm table to calculate more values of $\log(x)$, undergraduates investigate the determinant function on 2×2 invertible matrices. They find that, despite this function being a homomorphism, it cannot be used in the same way as $\log(x)$ to facilitate computation. To come to this conclusion, undergraduates analyze the work of a hypothetical classmate who is attempting to use the determinant function to compute matrix products; they then consider how they might help this student understand why his process will not work reliably. Ultimately, undergraduates establish that the bijectivity of $\log(x)$ is a key aspect of its ability to function as a computational aid.
   - *Discussion: Homomorphisms and Isomorphisms*
     The instructor motivates the formal definitions of both homomorphisms and isomorphisms and facilitates discussion to establish that isomorphic groups have the same algebraic structure. The instructor provides examples, such as the groups must have the same cardinality; if one group is commutative (resp. cyclic) so is the other; if one group has an element of order a, so must the other; etc.

    - *Problems 5–8*
      Undergraduates first identify the two groups between which the logarithm is an isomorphism. Then, they explain (with proof) why no isomorphism can exist between a series of groups by appealing to differences in algebraic structure.

3. Closure—Wrap-Up:
   The instructor concludes the lesson by reiterating that homomorphisms "preserve the operation" of a group and that two groups with an isomorphism between them have matching algebraic structures.

1.2 Alignment with College Curriculum

Isomorphisms and homomorphisms play a crucial role in the study of algebraic structures. Undergraduates explore these ideas by attempting to make computations by using the homomorphism property of the logarithm and determinant functions. The lesson introduces the formal mathematical definitions of isomorphism and homomorphism, also using the logarithm and determinant functions as preliminary examples. The lesson addresses that the existence of an isomorphism between two groups ensures that their algebraic structures will fully match (e.g., identical orders for corresponding elements, commutativity between corresponding elements is preserved, etc.).

1.3 Links to School Mathematics

The topic of logarithms is often presented in high school mathematics as a tool for solving exponential equations. While this is certainly a useful application of the function, a more complete treatment addresses that the logarithm predates formal exponential notation and was originally devised to help simplify tedious multiplication in the field of astronomy. The homomorphism property of logarithms is often treated as a fact to be memorized by secondary school students, but understanding the historical circumstances that gave rise to the logarithm helps ground this property in a meaningful context. The determinant, often computed in secondary school, is examined as a function on the set of invertible matrices and is discovered to be a group homomorphism.

This lesson highlights:

- The value of understanding and deriving results that may have been taken for granted, such as certain properties of logarithms.
- The role of knowing whether a function is injective, surjective, or bijective when comparing different functions.

This lesson addresses several mathematical knowledge and practice expectations included in high school standards documents, such as the Common Core State Standards for Mathematics (CCSSM, 2010). For example, high school students are expected to know how to graph and evaluate logarithmic functions using both analytical techniques and technology (c.f. CCSS.MATH.CONTENT.HSF.IF.C.7.e and CCSS.MATH.CONTENT.HSF.LE.A.4). To provide a point of comparison for the logarithm function this lesson also makes use of a determinant function on 2×2 matrices; high school students must also know of the matrix determinant (c.f. CCSS.MATH.CONTENT.HSM.VM.C.10) but also the basic operations of matrix arithmetic (c.f. CCSS.MATH.CONTENT.HSM.VM.C). Finally, this lesson emphasizes the need for viable mathematical arguments and encourages undergraduates to look for and make use of structural similarities.

1.4 Lesson Preparation

Prerequisite Knowledge

Undergraduates should know:

- The definition of a group. Some or all of the following related vocabulary is recommended:
    - Abelian group;
    - Cyclic group;
    - Order of a group (resp. an element);
- Basic familiarity with logarithms and determinants of 2×2 matrices;
- The definition of a bijection and how to tell if a function is bijective.

Learning Objectives

In this lesson, undergraduates will encounter ideas about teaching mathematics, as described in Chapter 1 (see the five types of connections to teaching listed in Table 1.2). In particular, by the end of the lesson undergraduates will be able to:

- Define group homomorphism and group isomorphism and identify key characteristics of each;
- Provide examples of group homomorphisms and group isomorphisms, as well as identify when a group homomorphism is or is not a group isomorphism;
- Use algebraic structure to explain when two groups cannot be isomorphic;
- Analyze hypothetical student work and assess understandings of group homomorphisms and group isomorphisms;
- Pose guiding questions to help a hypothetical student investigate function properties that preserve algebraic structure.

Anticipated Length

One 75-minute class session.

Materials

The following materials are required for this lesson:

- Pre-Activity (assign as homework prior to Class Activity)
- Class Activity
- Homework Problems (assign at the end of the lesson)
- Assessment Problems (include on quiz or exam after the lesson)

All handouts for this lesson appear at the end of this lesson, and LaTeX files can be downloaded from INSERT URL HERE.

1.5 Instructor Notes and Lesson Annotations

Before the Lesson

Assign the Pre-Activity as homework to be completed in preparation for this lesson.

> *We recommend that you collect this Pre-Activity the day before the lesson so that you can review undergraduates' responses before you begin the Class Activity. This will help you determine if you need to spend additional time reviewing the solutions to the Pre-Activity with your undergraduates.*

Pre-Activity Review (10 minutes)

Introduce the Pre-Activity by discussing the following connection to teaching:

Discuss This Connection to Teaching

High school students often only learn about the logarithm function as the inverse of the exponential function. In the course of solving these problems, they often memorize the property $\log(x \cdot y) = \log(x) + \log(y)$ without a meaningful understanding of the significance of this property. Understanding that this property is why logarithms were invented helps justify its existence and lessens the burden of memorization by giving meaningful context.

Next, discuss the solutions to the Pre-Activity as needed: if you saw that most undergraduates completed each problem correctly, you do not need to spend much time reviewing the solutions. However, if there are common areas that need further attention, you can give undergraduates time to deepen their understanding before you start the Class Activity. To accomplish this, we recommend first allowing your class to compare their answers in small groups. Then, call them together to share their findings in a whole-class discussion. See Chapter 1 for guidance on facilitating group work and selecting and sequencing student work for use in whole-class discussion.

In this lesson, we make no distinction between logarithms of different bases. You may ask your class to assume that we are using the common log (i.e., $\log_{10}(x)$) throughout the lesson; if so, you might prompt them to consider whether each of their conclusions remains valid in other bases.

Pre-Activity Problem 1

At the beginning of the 17^{th} century, two different mathematicians were working to simplify tedious calculations in the field of astronomy (where numbers are often very large) by creating a function that would "translate" multiplication into addition. After all, it is much easier to add numbers like 177,320,045 and 9,317,032,566 by hand than it is to multiply them. This work by John Napier from Scotland and Joost Bürgi from Switzerland led to the creation of the function that we now call the logarithm. The precise nature of the logarithm has been refined over the years by other mathematicians, but its original use as a computational aid is exemplified by a familiar identity: for all positive real numbers x and y,

$$\log(x \cdot y) = \log(x) + \log(y)$$

It is useful to think of this property as "preserving an operation"; that is, the logarithm function preserves multiplication by translating it into addition.

1. Of course, most functions on the real numbers do not translate a multiplicative structure into an additive one.

   (a) Find a pair of positive real numbers x and y that demonstrates that $\sin(x \cdot y)$ need not equal $\sin(x) + \sin(y)$. Find a pair of non-zero values for which, by coincidence, this relationship does hold.

   Sample Response:
   Most pairs of positive real numbers x and y will not have this property. For example, for $x = \pi$ and $y = 2$, $\sin(x \cdot y) = \sin(2\pi) = 0$. But $\sin(x) + \sin(y) = \sin(\pi) + \sin(2) = \sin(2)$, which is clearly not equal to 0.
   Some pairs (x, y) that do work:

   - For $(x, y) = (\sqrt{\pi}, -\sqrt{\pi})$, we first have $\sin(x \cdot y) = \sin(-\pi) = 0$. Then, we apply the fact that the sine function is odd: $\sin(x) + \sin(y) = \sin(\sqrt{\pi}) + \sin(-\sqrt{\pi}) = \sin(\sqrt{\pi}) - \sin(\sqrt{\pi}) = 0$.
   - If we let $x = \pi$, then $\sin(x \cdot y) = \sin(x) + \sin(y) \Rightarrow \sin(\pi \cdot y) = \sin(y) \Rightarrow \pi y = y + 2k\pi$, where $k \in \mathbb{Z}$. Solving for y yields a set of solutions: $\{(\pi, \frac{2k\pi}{\pi - 1}) : k \in \mathbb{Z}\}$.

   (b) Find a pair of positive real numbers x and y that demonstrates that $\sqrt{x \cdot y}$ need not equal $\sqrt{x} + \sqrt{y}$. Find a pair of non-zero values for which, by coincidence, this relationship does hold.

   Sample Response:
   Most pairs of positive real numbers x and y will not have this property. For example, for $x = 1$ and $y = 4$, $\sqrt{x \cdot y} = \sqrt{4} = 2$. But $\sqrt{x} + \sqrt{y} = \sqrt{1} + \sqrt{4} = 1 + 2 = 3$, which is clearly not equal to 2. On the other hand,

   - The pair $(4, 4)$ does work: $\sqrt{x \cdot y} = \sqrt{16} = 4 = 2 + 2 = \sqrt{4} + \sqrt{4} = \sqrt{x} + \sqrt{y}$.
   - If we let $x = k^2$ for some $k \in \mathbb{Z}$, then $\sqrt{x \cdot y} = \sqrt{x} + \sqrt{y} \Rightarrow |k|\sqrt{y} = |k| + \sqrt{y} \Rightarrow y = (\frac{|k|}{|k|-1})^2$. A set of solutions is thus $\{(k^2, (\frac{|k|}{|k|-1})^2) : k \in \mathbb{Z}\}$

   (c) Is there a pair of non-zero real numbers x and y for which $2(x \cdot y) = 2x + 2y$? If so, describe how to find all such pairs.

   Sample Response:
   Yes. Note that $2xy = 2x + 2y \Rightarrow xy = x + y \Rightarrow y = \frac{x}{x-1}$. Any pair $(x, \frac{x}{x-1})$ is a solution; for example, $(2, 2)$.

Commentary:
If your class is already comfortable with their solutions to these problems and you would like to initiate a more formal, abstract discussion (rather than an empirical one), you could use the following questions:

- Suppose f is a function on the real numbers such that $f(x \cdot y) = f(x) + f(y)$ for all x and y. What can you say about the value of $f(1)$?
- How can we use this conclusion to argue that the functions in parts (a)–(c) cannot preserve multiplication in the same way as the logarithm function?

With one minor additional hypothesis, it turns out that the logarithm is in fact the unique group homomorphism between $(\mathbb{R}^+, \cdot)$ and $(\mathbb{R}, +)$. See Dieudonne (1960, p. 82). Lastly, for further details on the history of logarithms, see Clark and Montelle (2011).

Problem 1 serves two important purposes: first, it establishes that a homomorphism must preserve the operation between any two arbitrary elements of a set; that is, undergraduates see that even trigonometric functions can be homomorphic on a subset of their domain. Second, it shows that there really was a need to "invent" a new function to translate multiplication into addition, since no extant functions at the time achieved this goal.

Pre-Activity Problems 2 & 3

2. A classmate claims that if everyone collectively forgot how to multiply two numbers together, logarithms would be useful for overcoming the memory lapse. That is, logarithms would be useful for computing products. Describe how you might still be able to compute the product $2 \cdot 3$ using the table of values for the logarithm function, given below, and the identity $\log(x \cdot y) = \log(x) + \log(y)$.

x	1	2	3	4	5	6	7	8	9
$\log(x)$	0	.301	.477	.602	.699	.778	.845	.903	.954

Sample Response:
I can use the chart to see that $\log(2) = .301$ and $\log(3) = .477$. These values added together are .778, which we see is the value for $\log(6)$. Since $\log(2 \cdot 3) = \log(2) + \log(3) = \log(6)$, and because we know the logarithm function is bijective (and thus injective), we conclude $2 \cdot 3 = 6$.

3. Another classmate claims that since 72 is the product of 8 and 9, the table can be used to calculate $\log(72)$. They claim that this is possible even though 72 is not an entry in the first row of the table. Moreover, they claim that the table can be used to compute the logarithm of any number which is the product of numbers represented in the first row of the table. Use the table and the identity $\log(x \cdot y) = \log(x) + \log(y)$ to:

(a) Calculate $\log(15)$.

Solution:
Since $15 = 5 \cdot 3$, we have $\log(15) = \log(5 \cdot 3) = \log(5) + \log(3) = .699 + .477 = 1.176$.

(b) Calculate $\log(24)$ in two different ways.

Solution(s):

- Since $24 = 8 \cdot 3$, we have $\log(24) = \log(8 \cdot 3) = \log(8) + \log(3) = .903 + .477 = 1.380$.
- Since $24 = 6 \cdot 4$, we have $\log(24) = \log(6 \cdot 4) = \log(6) + \log(4) = .778 + .602 = 1.380$.

(c) Estimate $\log(17)$. Why isn't it possible to calculate $\log(17)$ precisely?

Sample Response:
Since 17 is prime, we cannot factor it and use the chart. Instead, we might calculate $\log(16) = \log(4) + \log(4) = 1.204$ and $\log(18) = \log(9) + \log(2) = 1.255$ and then average them. By this method of estimation, $\log(17) \approx 1.230$.

Commentary:
Consider using some of the following questions to engage groups who finish quickly:

- Is it true that $\log(x \cdot y \cdot z) = \log(x) + \log(y) + \log(z)$? How do you know?
- What's the fewest number of entries on a logarithm chart such as this one that you would need to calculate the log values of all the integers up to 20? 100?
- How precise is your answer to Problem 3(c)? How could you make it more accurate?

To transition to the Class Activity, ask one or more groups to share their solution to Problem 2. Allow the other groups to comment on whether they agree or disagree with the procedure. Tell the class that in today's activity they will compare and contrast the logarithm with other functions that "preserve an operation" to better see what additional properties the logarithm has that make it especially appropriate for these kinds of calculations.

Class Activity: Problems 1–4 (30 minutes)

Prepare undergraduates for the Class Activity by asking them to brainstorm everything they know about logarithms and logarithmic functions. Write all undergraduate answers on the board—if necessary, discuss discrepancies and highlight correct and precise responses. Some possible undergraduate responses are listed below; if it is not supplied by your class, make sure that you lead them to generate the first item (possibly by using the second or third items).

- Bijective
- Injective/ Surjective
- Inverse of the exponential function
- Monotonically increasing
- Concave down
- The natural logarithm has base e and is written $\ln(x)$; the common logarithm has base 10 and is written $\log(x)$.
- The graph of the logarithm function is stretched or shrunk vertically by changing the base of the logarithm.
- $\log(1) = 0$; $\log(x) < 0$ for $x < 1$ and $\log(x) > 0$ for $x > 1$

Leave this list on the board. Undergraduates will need to refer to it in Problem 4.

Distribute the Class Activity. Allow your class to work in small groups on Problem 1. As you monitor their work in groups, you might use some of the following questions to prompt discussion:

- How might you use $\log(x \cdot y) = \log(x) + \log(y)$ to explain why it makes sense that the values of $\log(x)$ are negative when $x < 1$?
- What difficulties did you encounter when attempting to find the product of 1.2 and 1.2 using this chart? How did you overcome them?

For further details on the use of logarithms by NASA in the Apollo program, see Nadworny (2014).

Class Activity Problem 1

The steps used by astronomers and other scientists to calculate the product $x \cdot y$ using a logarithm function looked like the following:

- First, calculate $\log(x)$ and $\log(y)$.
- Next, add these values together.
- Finally, find a positive real number whose logarithm is the sum $\log(x) + \log(y)$. This number is $x \cdot y$.

For the first and last step, charts called log tables were created which recorded, as efficiently as possible, $\log(x)$ for all real numbers x. A (simplified) log table is included below.

	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	n/a	-1.000	-0.699	-0.523	-0.398	-0.301	-0.222	-0.155	-0.097	-0.046
1	0.000	0.041	0.079	0.114	0.146	0.176	0.204	0.230	0.255	0.279
2	0.301	0.322	0.342	0.362	0.380	0.398	0.415	0.431	0.447	0.462
3	0.477	0.491	0.505	0.519	0.531	0.544	0.556	0.568	0.580	0.591
4	0.602	0.613	0.623	0.633	0.643	0.653	0.663	0.672	0.681	0.690
5	0.699	0.708	0.716	0.724	0.732	0.740	0.748	0.756	0.763	0.771
6	0.778	0.785	0.792	0.799	0.806	0.813	0.820	0.826	0.833	0.839
7	0.845	0.851	0.857	0.863	0.869	0.875	0.881	0.886	0.892	0.898
8	0.903	0.908	0.914	0.919	0.924	0.929	0.934	0.940	0.944	0.949
9	0.954	0.959	0.964	0.968	0.973	0.978	0.982	0.987	0.991	0.996

For example, $\log(1.3)$ is found at the intersection of the row marked "1" and the column marked ".3", so $\log(1.3) = 0.114$. Log tables (and a related tool, the slide rule) were in use well into the 1900s, and were even used by NASA to make calculations for the Apollo 11 moon landing.

1. Use the above process and the log table provided to find the product of the following numbers:

   (a) 1.5 and 2.0

   Solution:
   $\log(1.5) + \log(2.0) = 0.176 + 0.301 = 0.477 \Rightarrow 1.5 \cdot 2.0 = 3.0$

   (b) 0.5 and 8.6

   Solution:
   $\log(0.5) + \log(8.6) = -0.301 + 0.934 = 0.633 \Rightarrow 0.5 \cdot 8.6 = 4.3$

   (c) 1.2 and 1.2

   Sample Response:
   $\log(1.2) + \log(1.2) = 0.079 + 0.079 = 0.158 \Rightarrow 1.2 \cdot 1.2 \approx 1.4$

Once you have seen that most groups have finished Problem 1, initiate a classroom discussion in which you ask groups to describe their process for 1(c). Because 0.158 is not on the chart, undergraduates may have different strategies for finding the value of 1.2^2. For example, undergraduates may say that:

- 0.158 is closer to 0.146 than 0.176, so $1.2^2 \approx 1.4$.
- Because $0.176 - 0.146 = 0.030$ and $0.158 - 0.146 = 0.012$, we can linearly approximate the value of 1.2^2 as $1.4 + \frac{0.012}{0.030} \times 0.1 = 1.4 + 0.04 = 1.44$.

Afterward, allow your class to work in small groups on Problem 2. If your class is unfamiliar with linear algebra concepts, before giving them time to work on this task you might choose to first remind them how to find the determinant of a 2×2 matrix and/ or how to tell if a 2×2 matrix is invertible using the determinant.

Class Activity Problem 2

2. Another setting in advanced mathematics where a function "preserves" a difficult operation is found in linear algebra. Consider the set $GL_2(\mathbb{R}) = \{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] : a, b, c, d \in \mathbb{R}, ad - bc \neq 0\}$, which consists of all invertible 2×2 matrices with real number entries, and the determinant function $det : GL_2(\mathbb{R}) \to \mathbb{R}$,

which assigns to each matrix its determinant.

Recall from linear algebra that for any two matrices M and N in $GL_2(\mathbb{R})$, we have that $det(\mathrm{MN}) = det(\mathrm{M}) \cdot det(\mathrm{N})$. Choose two 2×2 matrices that you know are invertible and verify this identity holds for your two matrices.

Solution:
Answers will vary. Undergraduate responses should largely mirror the following example:

$$\begin{aligned} det(\mathrm{MN}) &= det(\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right]) \\ &= det(\left[\begin{smallmatrix} 2 & 1 \\ 0 & 1 \end{smallmatrix}\right]) \\ &= 2 \\ &= 1 \cdot 2 \\ &= det(\mathrm{M}) \cdot det(\mathrm{N}) \end{aligned}$$

When you call the class back together to compare answers, instead of asking each group to report out individually, ask whether any group found a pair of matrices for which this identity did NOT appear to hold. Address their proposed counterexample on the board. If instead everyone agrees that this identity should hold for any two matrices, you may choose not to do your own example on the board in the interest of time. Let undergraduates work in small groups on the next two problems simultaneously. Make sure to circulate the classroom to observe their progress.

Class Activity Problems 3 & 4

3. Finn hopes that the determinant function (see previous question) can help them to avoid multiplying matrices, much like the logarithm helped astronomers avoid multiplication of large, real numbers. Finn models the following strategy based on the process given before Problem 1:

   - First, calculate $det(\mathrm{M})$ and $det(\mathrm{N})$
   - Next, multiply these values together.
   - Finally, find a matrix whose determinant is the product $det(\mathrm{M}) \cdot det(\mathrm{N})$. This matrix is MN.

   (a) Why won't Finn's strategy work?

   Solution:
   At the last step, there are multiple matrices that have the same determinant. Simply finding ONE matrix whose determinant is the same as the product $det(\mathrm{M}) \cdot det(\mathrm{N})$ doesn't guarantee that this matrix is the product MN.

   (b) What question might you ask Finn to help them see the flaw in this plan? Why do you think this question would be helpful?

   Sample Responses:

   - Can two matrices have the same determinant? This question would help Finn see that a matrix is not uniquely determined by its determinant.
   - Is the determinant function injective? Why is this property important? This sequence of questions would help Finn consider whether det has an inverse function that would allow their procedure to work.
   - What is the determinant of $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 4 \end{smallmatrix}\right]$? What about $\left[\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right]$? By asking Finn to calculate the determinant of two different matrices, they will see that the determinant function cannot be injective and that the last step of the process isn't reasonable.

4. What is a key property that the logarithm function applied to positive real numbers possesses that the determinant function applied to invertible matrices does NOT possess?

   Solution:
   The key idea is that the logarithm function is bijective, but that the determinant function is not. Another sample response: "You can have two matrices with the same determinant so the determinant function isn't one-to-one."

Commentary:
As you monitor the undergraduates during their group work,

- If undergraduates are having trouble on Problem 3, encourage them to attempt to apply Finn's strategy to the matrices M and N which they chose in Problem 2.
- Ask undergraduates to refer to the list of properties of the logarithm function, which should still be on the board, when they reach Problem 4.

Questions you might use to prompt discussion:

- How can we use $det(\mathrm{MN}) = det(\mathrm{M}) \cdot det(\mathrm{N})$ to describe the relationship between the determinant of a matrix and the determinant of its inverse?
- Do you think the homomorphism property of the determinant function still holds for non-invertible matrices? Why or why not?

Call the class back together and allow volunteers to report out on Problems 3(a) and 4 especially. Point out that, while the determinant map considered here still "preserves an operation" like the logarithm, because it is not also bijective it cannot be used to "avoid" the original operation: in this case, matrix multiplication.

Before transitioning to the next portion of the lesson, discuss the following connection to teaching with your class.

Discuss This Connection to Teaching

Problems such as Problem 3(b) are important because it is useful for all undergraduates to think about how others use and reason with mathematics. Posing questions gives undergraduates the opportunity to think about how they would respond to another person's work in ways that would help that person develop an understanding of the concept, which improves both their own understanding of the content and their ability to communicate technical mathematics.

Discussion: Homomorphisms and Isomorphisms (10 minutes)

To bridge to the next class activity, take time to have undergraduates understand that the logarithm map is a bijection, but the determinant map is not. This key difference can be one way to motivate the definitions of homomorphisms and isomorphisms of groups. This can be orchestrated by sequencing student work from the previous problems or other ways that the instructor feels are appropriate for their context. Use the notation from your own notes or your own textbook, but emphasize or highlight the following:

- Homomorphisms "preserve the operation"; that is, one can do the group operation either in the domain (before applying the homomorphism) or in the codomain (after applying the homomorphism) and expect the same result.
- Because they both "preserve the operation," the image of a group homomorphism (and hence, a group isomorphism) is itself a group. (A proof of this is included as a sequence of homework exercises.)
- Isomorphisms, since they are bijections, also preserve the cardinality of the sets.

At this point in the class discussion, discuss the following connection to teaching.

Discuss This Connection to Teaching

High school students often use the inverse relationship between exponents and logarithms to solve problems involving exponents and logarithms. Prospective teachers can leverage their understanding that the logarithm map is a bijection to discuss why it is appropriate to apply logarithms when solving exponential equations or use exponential functions when solving logarithmic equations.

After introducing definitions of homomorphisms and isomorphisms of groups, emphasize that one of the many uses of an isomorphism is that isomorphic groups share the same structural characteristics. Consider using simpler language first to provide some of the following illustrative examples of such characteristics: If G and H are isomorphic, then

- G and H have the same number of elements (i.e., the same cardinality).
- If the elements of G commute, then so do the elements of H.
- If G has an element g such that $g^n = e_G$, then H has an element h such that $h^n = e_H$.
- If G is generated by a single element, so is H.
- If every element of G is its own inverse, every element of H is its own inverse.

If undergraduates are already familiar with any of the key vocabulary necessary (order of an element, cyclic, abelian, etc.) to list these properties with mathematical precision, you may instead choose to write them formally. Proofs of these properties are included in the additional homework section of this document. Another useful analogy, if your undergraduates are comfortable working with Cayley tables, is that an isomorphism between two groups acts as a "key" that lets you easily fill in one table assuming you already know what the other looks like. The Klein 4-group and $\mathbb{Z}_2 \times \mathbb{Z}_2$ illustrate this concept well.

Class Activity: Problems 5–8 (20 minutes)

Problem 5 can be asked and answered at the front of the room without requiring the undergraduates to consult each other in groups. Make sure that undergraduates are specific about the group operations of the domain and codomain. If you feel it would benefit your class, you might also ask: "Between which two groups is the determinant function a homomorphism?"

Class Activity Problem 5

5. Between which two groups is the logarithm function an isomorphism?

   Solution:
   The logarithm is an isomorphism from $(\mathbb{R}^+, \cdot)$ to $(\mathbb{R}, +)$.

Use Problem 5 to discuss the following connection to teaching before moving on to the remainder of the Class Activity:

Discuss This Connection to Teaching

In mathematics, it is important to use precise language in order to avoid ambiguity. When we name a group, we are sure to include its operation; when we claim a function is an isomorphism, it must be an isomorphism between two spaces. Secondary school teachers must also take steps to ensure that both they and their students use mathematical language in an unambiguous way. For example, in geometry, students often say that two objects (such as triangles) are "the same." In this case, teachers who have developed sensitivity for precise language may prompt their students to be more specific. That is, are the triangles "the same" because they are congruent or because they are similar? Or are they the same type of triangle (e.g., obtuse, right) but otherwise of a different shape?

Allow undergraduates to work in small groups on Problems 6, 7, and 8 simultaneously. If appropriate, you may want to discuss Problem 6 as a class first. Consider using the following sequence of prompts to help motivate this discussion:

- Use our list to describe the group structure of $(GL_2(\mathbb{R}), \cdot)$ and $(\mathbb{R}^*, \cdot)$.
- What would we know about the structure of the two groups if there were a group isomorphism between them?

As undergraduates work, walk around the classroom and check for interesting approaches and help correct misunderstandings. Also, as you monitor your class,

- Encourage groups that finish quickly to look for more than one structural difference between the groups in each question.
- Where appropriate, ask groups to justify their claims about the structure of the groups in these problems, at least informally. For example:
  - If $(\mathbb{Q}^+, \cdot)$ were cyclic, what would its generator be? Why doesn't this make sense?
  - Is there a bijection between $\mathbb{R}$ and $\mathbb{Z}$ when they are viewed just as sets?

Class Activity Problems 6, 7 & 8

6. Now that Finn understands that the function det is not a group isomorphism and will not help avoid matrix multiplication, they want to try and find another function from $(GL_2(\mathbb{R}), \cdot) \to (\mathbb{R}^*, \cdot)$ which is a group isomorphism, where $\mathbb{R}^*$ denotes the set of nonzero real numbers. Does such a function exist? Why or why not?

   Solution:
   No. $(GL_2(\mathbb{R}), \cdot)$ is not an abelian group, but $(\mathbb{R}^*, \cdot)$ is. Since the structure of the two groups is different, there can be no isomorphism between them.

7. Finn shifts focus to finding a "better logarithm"; that is, a group isomorphism from $(\mathbb{R}^+, \cdot) \to (\mathbb{Z}, +)$ that allows them to multiply positive real numbers by adding integers instead of real numbers. Does such a function exist? Why or why not?

   Solutions:

   - No. $\mathbb{R}^+$ is an uncountable set, but $\mathbb{Z}$ is countable. Since the two groups have different orders, there can be no isomorphism between them.
   - No. $(\mathbb{Z}, +) = \langle 1 \rangle$, while $(\mathbb{R}^+, \cdot)$ is not cyclic (it is uncountable and all cyclic groups are at most countable). Since the two groups have different algebraic structures, there can be no isomorphism between them.

8. Maybe, Finn says, we could at least find a group isomorphism from $(\mathbb{Q}^+, \cdot) \to (\mathbb{Z}, +)$ that lets us multiply positive rational numbers by adding integers. Does such a function exist? Why or why not?

   Solution:
   No. $(\mathbb{Z}, +) = \langle 1 \rangle$, but $(\mathbb{Q}^+, \cdot)$ is not cyclic. Since the structure of the two groups is different, there can be no isomorphism between them.

Wrap-Up (5 minutes)

Recap the lesson briefly for the class:

- Homomorphisms "preserve the operation" of a group. Consequently, the image of a group homomorphism (and hence, a group isomorphism) is itself a group.
- Two groups with an isomorphism between them have matching algebraic structures (e.g., identical orders for corresponding elements, commutativity between corresponding elements is preserved, etc.)

Emphasize that, as we saw with the logarithm function, isomorphisms also allow one to avoid working in a particular group with a difficult operation by mapping elements into the codomain group and working with its operation before mapping back. In this way, we can understand everything essential about a group by studying a different group to which it is isomorphic.

Discuss This Connection to Teaching

Students are familiar with the fact that linear functions of the form $h(x) = cx$, where $c \in \mathbb{R}$, preserve structure. For example, if $f(x) = 2x$, we have $f(a+b) = 2(a+b) = 2a+2b = f(a)+f(b)$. However, they often overgeneralize this understanding when working with $g(x) = x^2$ even though $g(a+b) = (a+b)^2 \neq a^2 + b^2 = g(a) + g(b)$. Prospective teachers' advanced understanding of group isomorphisms supports their capacity to discuss how the logarithm function, say, "converts" multiplication problems into addition problems which can also be a common way to motivate the technique of logarithmic differentiation in calculus.

We end the lesson using an exit ticket. See Chapter 1 for guidance about how to conclude mathematics lessons with exit tickets.

Homework Problems

At the end of the lesson, assign the following homework problems.

Undergraduates need to know that the fact that homomorphisms preserve the operation means that they also preserve some aspects of the structure of a group. Problem 1 illustrates that the image of homomorphism acting on a group will be a subgroup of the codomain.

Homework Problem 1

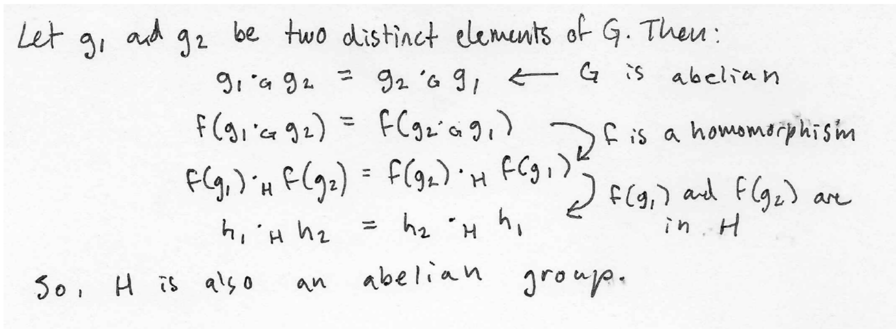
1. Let $(G, \cdot_G)$ and $(H, \cdot_H)$ be groups and $f : G \to H$ be a homomorphism.
   (a) Show that the image of the group identity of G under f must be the group identity of H (that is, $f(e_G) = e_H$).
   Solution:
   First, note that $f(e_G) = f(e_G \cdot_G e_G)$. Because f is a homomorphism, we may rewrite this equation as $f(e_G) = f(e_G) \cdot_H f(e_G)$. By definition, then, $f(e_G) = e_H$.
   (b) Use this result to show that, given an arbitrary element a in G, the image of the inverse of a under f must be the inverse of the image of a under f (that is, $\forall a \in G$, $f(a^{-1}) = f(a)^{-1}$).
   Solution:
   Let $a \in G$.Using the result from 1(a) and the fact that f is a homomorphism, $e_H = f(e_G) = f(a \cdot_G a^{-1}) = f(a) \cdot_H f(a^{-1})$. Similar work shows that $e_H = f(a^{-1}) \cdot_H f(a)$. Then, by the uniqueness of a group inverse, we have that $f(a^{-1}) = f(a)^{-1}$, as desired.
   (c) Use the previous two results to show that the image of G under f is itself a group with the operation of H.
   Solution:
   - Closure follows directly from the fact that f is a homomorphism: Let $f(a), f(b) \in f(G)$. Then, $f(a) \cdot_H f(b) = f(a \cdot_G b) \in f(G)$.
   - Associativity of $f(G)$ is inherited from associativity of H.
   - By 1(a), $e_H = f(e_G) \in f(G)$.
   - By 1(b), an element $f(a) \in f(G)$ has inverse $f(a^{-1})$

   Thus, $f(G)$ is a group under the same operation as H.

In Problem 2, undergraduates prove one of the claims made during the discussion of the Class Activity. Before doing so, they analyze a hypothetical student's attempt at a proof and identify an error; importantly, they also evaluate how the given proof might not be incorrect in a different context.

Homework Problem 2

2. Let $(G, \cdot_G)$ and $(H, \cdot_H)$ be groups and $f : G \to H$ be a homomorphism. Furthermore, let G be an abelian group. Anjali writes the following proof, which she claims shows that H is also an abelian group:



Let g_1 and g_2 be two distinct elements of G. Then:

$g_1 \cdot_G g_2 = g_2 \cdot_G g_1$ ← G is abelian

$f(g_1 \cdot_G g_2) = f(g_2 \cdot_G g_1)$ } f is a homomorphism

$f(g_1) \cdot_H f(g_2) = f(g_2) \cdot_H f(g_1)$ } $f(g_1)$ and $f(g_2)$ are in H

$h_1 \cdot_H h_2 = h_2 \cdot_H h_1$

So, H is also an abelian group.

(a) Explain why Anjali's proof does not show that H is an abelian group. What has she proven instead?

Solution:
At the last line, h_1 and h_2 are not arbitrary elements of H; in fact, they may not even be distinct elements. Anjali has instead proven that elements in the image of G under f commute with each other.

(b) Given the same groups $(G, \cdot_G)$ and $(H, \cdot_H)$ let $f : G \to H$ now be an **isomorphism**. Add to Anjali's proof to show that, under these conditions, H is now an abelian group.

Solution:
Before claiming that H is abelian, Anjali should add: "Because f is injective, we know that h_1 and h_2 are two distinct elements of H. Because f is also surjective, we know that every element of H is in the image of f; thus, h_1 and h_2 also represent arbitrary elements of H."

Problem 3 helps undergraduates appreciate that saying "the logarithm is a homomorphism" is not sufficiently specific; one needs to specify the groups and operations involved. Furthermore, it gives undergraduates the opportunity to explore how they might translate their advanced mathematical understanding of logarithms into language appropriate for a high school student.

Homework Problem 3

3. Imagine that you are a high school mathematics teacher who has noticed that one of your students, Hai, has assumed that $\log(x + y) = \log(x) + \log(y)$ when simplifying logarithmic expressions on a homework assignment.

(a) Find a pair of real numbers x and y demonstrating that Hai's assumption is not always true. For which pairs of positive real numbers (x, y) is Hai's assumption true?

Sample Response:
For $x = 1$ and $y = 2$, $\log(x + y) = \log(3)$ but $\log(x) + \log(y) = \log(1) + \log(2) = \log(2)$. Clearly, $\log(3) \neq \log(2)$. More generally, choosing $x = 1$ and $y = a$ will yield $\log(1 + a) =$

$\log(a)$, which can never be true given that the logarithm function is a bijection. Then:

$$\log(x+y) = \log(x) + \log(y)$$
$$10^{\log(x+y)} = 10^{\log(x)+\log(y)} = 10^{\log(x)} \cdot 10^{\log(y)}$$
$$x + y = x \cdot y$$
$$y = \frac{x}{x-1}$$

So any pair of real numbers (x, y) with $x, y > 0$ satisfying $y = \frac{x}{x-1}$ will also satisfy $\log(x+y) = \log(x) + \log(y)$.

(b) Using what you have learned about logarithms in this lesson, how might you help Hai understand that $\log(x \cdot y) = \log(x) + \log(y)$ is the correct identity? Why is your explanation helpful? Make sure your explanation is appropriate for a high school student.

Sample Responses:

- I could explain that the logarithm was created in order to help astronomers turn multiplication problems into addition problems. That means that one side of the identity needs to feature multiplication as an operation somewhere.
- I know that the logarithm function is a group isomorphism, so it should be injective. I could show Hai my work for part (a) to demonstrate that his identity leads to a logarithm function that is NOT injective; then, I might show him that choosing $x = 1$ in $\log(x \cdot y) = \log(x) + \log(y)$ does not create such a problem.

The familiar additive group of integers modulo n is shown to be related to the additive group of integers by a natural mapping. In Problem 4(b), undergraduates need to both argue from the definition of an isomorphism and from the concept of an isomorphism as structure-preserving to make their point. This demonstrates a thorough and versatile understanding of the material.

Homework Problem 4

4. Recall that $\mathbb{Z}_n$ is the set of *equivalence classes* on the integers, where two integers are in the same equivalence class if and only if they both have the same (smallest, non-negative) remainder when divided by n. The set $\mathbb{Z}_n$ contains n such equivalence classes which, canonically, are represented by the possible remainders when an integer is divided by n: $\{0, 1, \ldots, n-2, n-1\}$.

(a) Let θ be the map from $(\mathbb{Z}, +)$ to $(\mathbb{Z}_n, +)$ given by $\theta(z) = r$, where z is an integer and r is its remainder when divided by n. Show that θ is a group homomorphism. You may assume without proof that $(\mathbb{Z}, +)$ and $(\mathbb{Z}_n, +)$ are in fact groups.

Solution:

Let $a, b \in \mathbb{Z}$. Then by the division algorithm, $\exists\, q_1, q_2, r_1, r_2 \in \mathbb{Z}$ such that $a = q_1 \cdot n + r_1$ and $b = q_2 \cdot n + r_2$, where $0 < r_1, r_2 < n$. Note that $\theta(a+b) = \theta((q_1+q_2)\cdot n + r_1 + r_2)$. If $r_1 + r_2 < n$, then $\theta((q_1+q_2)\cdot n + r_1 + r_2) = r_1 + r_2 = \theta(a) + \theta(b)$, as desired.

If $r_1 + r_2 \geq n$, then $\exists\, q_3, r_3 \in \mathbb{Z}$ such that $r_1 + r_2 = q_3 \cdot n + r_3$, where $0 < r_3 < n$. Then, $\theta((q_1+q_2)\cdot n + r_1 + r_2) = \theta((q_1+q_2+q_3)\cdot n + r_3) = r_3$. Note that here, r_3 represents the equivalence class containing those integers that differ from r_3 by a multiple of n. In particular, this equivalence class includes the element $r_3 + q_3 \cdot n = r_1 + r_2 = \theta(a) + \theta(b)$. So, $\theta(a+b) = \theta(a) + \theta(b)$, as desired.

(b) Explain why θ cannot be an isomorphism in two ways: by showing that θ is not a bijection and by finding a difference in the algebraic structures of $(\mathbb{Z}, +)$ and $(\mathbb{Z}_n, +)$ (other than that their orders differ).

Solution:

- θ is not injective: $\theta(n) = \theta(2n) = 0$, but $n \neq 2n$ given that $n > 1$.
- The order of the elements are also different: for any nonzero $z \in (\mathbb{Z}, +)$, $|z| = \infty$; however, every nonzero element of $(\mathbb{Z}_n, +)$ has finite order.

Problem 5 accomplishes two goals: it ties important calculus knowledge to the abstract algebra curriculum and, in part (b), it helps undergraduates informally verify that the kernel of a homomorphism is always a subgroup of the domain.

Homework Problem 5

5. Let F be the set of continuous, real-valued functions on the interval $[0, 1]$. Let σ be the map from $(F, +)$ to $(\mathbb{R}, +)$ given by $\sigma(f) = \int_0^1 f(x)dx$ for all f in F. The operation "+" on F is defined by $(f + g)(x) = f(x) + g(x)$ for all f, g in F and $x \in [0, 1]$. The operation "+" on $\mathbb{R}$ is the usual real number addition.

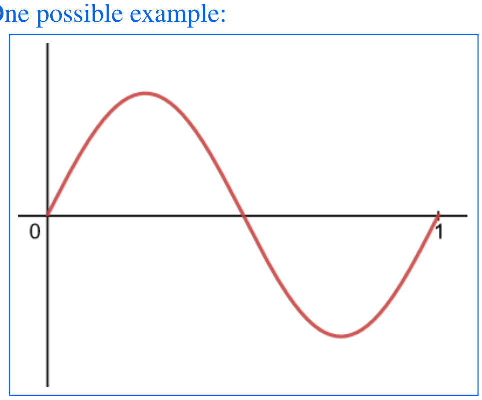
(a) Show that σ is a group homomorphism. You may assume without proof that $(F, +)$ and $(\mathbb{R}, +)$ are in fact groups.

Solution:
Let $f, g \in F$. Then: $\sigma(f + g) = \int_0^1 (f + g)(x)dx = \int_0^1 [f(x) + g(x)]dx = \int_0^1 f(x)dx + \int_0^1 g(x)dx = \sigma(f) + \sigma(g)$, as desired.

(b) Describe the function that represents the identity element in $(F, +)$. Now, choose a different element of $(F, +)$ which maps to the identity element of $(\mathbb{R}, +)$ under the map σ. Draw a graph of your chosen function and explain how you know it meets this criteria.

Solution:

- The function that acts as the identity element in $(F, +)$ is the function whose values are equal to 0 for every $x \in [0, 1]$.
- Undergraduate graphs will vary, but the integral of each graphed function over the interval $[0, 1]$ should be zero. One possible example:



- Note that the identity of $(\mathbb{R}, +)$ is 0. If for $f \in F$ we have that $\sigma(f) = 0$, then the area bounded between f and the x-axis is the same both above and below the x-axis.

(c) Let $ker(\sigma)$ be the set of **all** elements of $(F, +)$ that map to the identity element of $(\mathbb{R}, +)$ under the map σ. Show that $ker(\sigma)$ is a subgroup of $(F, +)$.

Solution:

- Let $f, g \in ker(\sigma)$. Then $\sigma(f+g) = \sigma(f) + \sigma(g) = 0 + 0 = 0$, so $ker(\sigma)$ is closed.
- $ker(\sigma)$ inherits associativity from F.
- Let e be the zero function given by $e(x) = 0$ for all $x \in [0, 1]$. This is the identity element of $(F, +)$ described in part (a). Then, $\sigma(e) = 0$, so $e \in ker(\sigma)$.
- Let $f \in ker(\sigma)$. Since $f(x) + (-f)(x) = (f - f)(x) = e(x)$ for all $x \in [0, 1]$, the function $-f$ is the inverse of f. To see that $-f$ must also be in $ker(\sigma)$, note that $\sigma(-f) = \int_0^1 (-f)(x)dx = -\int_0^1 (f)(x)dx = 0$.

Thus, $ker(\sigma)$ is a group.

Assessment Problems

The following two problems address ideas explored in the lesson, with a focus on connections to teaching and mathematical content. You can include these problems as part of your usual course quizzes or exams.

Assessment Problems 1 & 2

1. Let $M_2(\mathbb{R}) = \{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] : a, b, c, d \in \mathbb{R}\}$ and define the function $tr : (M_2(\mathbb{R}), +) \to (\mathbb{R}, +)$ to be the map $tr(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]) = a + d$. That is, tr assigns to each matrix its trace.

   (a) Show that tr is a group homomorphism. You may assume without proof that $(M_2(\mathbb{R}), +)$ and $(\mathbb{R}, +)$ are in fact groups.

   Solution:

   $$\begin{aligned} tr(\mathrm{M}+\mathrm{N}) &= tr(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] + \left[\begin{smallmatrix} e & f \\ g & h \end{smallmatrix}\right]) \\ &= tr(\left[\begin{smallmatrix} a+e & b+f \\ c+g & d+h \end{smallmatrix}\right]) \\ &= a + e + d + h \\ &= (a+d) + (e+h) \\ &= tr(\mathrm{M}) + tr(\mathrm{N}) \end{aligned}$$

   (b) Is tr an isomorphism? Why or why not?

   Solution:

   The function tr is not injective, and thus not a bijection. Two matrices can certainly have the same trace but represent fundamentally different linear operators:
   $tr([\begin{smallmatrix} 1 & 7 \\ 0 & 1 \end{smallmatrix}]) = 2$
   $tr([\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}]) = 2$

2. Abina claims that there must be an isomorphism between $(\mathbb{Z}_4, +)$ and $(\mathbb{Z}_2 \times \mathbb{Z}_2, +)$ since the orders of the groups are the same.

   (a) Based on her claim, what do you think Abina understands about isomorphisms?

   Sample Responses:

   - Abina knows that isomorphisms are bijections, and so both the domain and the codomain must have the same order.
   - Abina knows that isomorphisms preserves structure, e.g., order.
   - I think she understands that if two groups are isomorphic that they have the same order, but is thinking she can flip the "if, then" statement and say "If two groups have the same order, then they are isomorphic."

(b) Explain why the two groups in question cannot be isomorphic.

Solution(s):

- Every element of $(\mathbb{Z}_2 \times \mathbb{Z}_2, +)$ is its own inverse. However, the inverse of 1 in $(\mathbb{Z}_4, +)$ is 3 since $1 + 3 = 4 = 0$ (mod 4).
- While $(\mathbb{Z}_4, +)$ is cyclic, $(\mathbb{Z}_2 \times \mathbb{Z}_2, +)$ is not. The groups generated by the elements of $(\mathbb{Z}_2 \times \mathbb{Z}_2, +)$ are:

  $\langle(0,0)\rangle = \{(0,0)\}$
  $\langle(1,0)\rangle = \{(0,0),(1,0)\}$
  $\langle(0,1)\rangle = \{(0,0),(0,1)\}$
  $\langle(1,1)\rangle = \{(0,0),(1,1)\}$

  Since none of these is the whole group, $(\mathbb{Z}_2 \times \mathbb{Z}_2, +)$ is not cyclic.

(c) What question would you ask Abina to help her understand her mistake? Why do you think your question would be helpful?

Sample Responses:

- What is the difference between an isomorphism and a bijection? This question will help Abina realize that she is not attending to the entire definition of isomorphism.
- What is a homomorphism, and how is it different from an isomorphism? This question will help Abina realize that she is not attending to the entire definition of isomorphism.
- Is $(\mathbb{Z}_2 \times \mathbb{Z}_2, +)$ cyclic? This question will help Abina identify that the algebraic structure of the groups differ in a key way.
- What is the order of each element of $(\mathbb{Z}_2 \times \mathbb{Z}_2, +)$? Of $(\mathbb{Z}_4, +)$? This question will help Abina identify that the algebraic structure of the groups differ in a key way.
- We know that $(\mathbb{Z}, +)$ and $(\mathbb{Q}^+, \cdot)$ have the same order. What is different about the structure of these two groups? This question will prompt Abina to compare the current situation to a similar problem from the class activity.

1.6 References

[1] Clark, K. M. & Montelle, C. (2011) *Logarithms: The early history of a familiar function*. Retrieved from `https://www.maa.org/press/periodicals/convergence/logarithms-the-early-history-of-a-familiar-function-introduction`

[2] National Governors Association Center for Best Practices & Council of Chief State School Officers (2010). *Common Core State Standards for Mathematics.* Authors. Retrieved from `http://www.corestandards.org/`

1.7 Lesson Handouts

Handouts for use during instruction are included on the pages that follow. LaTeX files for these handouts can be downloaded from INSERT URL HERE.

NAME: **PRE-ACTIVITY:** LOGARITHMS AND ISOMORPHISMS (page 1 of 2)

At the beginning of the 17^{th} century, two different mathematicians were working to simplify tedious calculations in the field of astronomy (where numbers are often very large) by creating a function that would "translate" multiplication into addition. After all, it is much easier to add numbers like 177,320,045 and 9,317,032,566 by hand than it is to multiply them. This work by John Napier from Scotland and Joost Bürgi from Switzerland led to the creation of the function that we now call the logarithm. The precise nature of the logarithm has been refined over the years by other mathematicians, but its original use as a computational aid is exemplified by a familiar identity: for all positive real numbers x and y,

$$\log(x \cdot y) = \log(x) + \log(y)$$

It is useful to think of this property as "preserving an operation"; that is, the logarithm function preserves multiplication by translating it into addition.

1. Of course, most functions on the real numbers do not translate a multiplicative structure into an additive one.

   (a) Find a pair of positive real numbers x and y that demonstrates that $\sin(x \cdot y)$ need not equal $\sin(x) + \sin(y)$. Find a pair of non-zero values for which, by coincidence, this relationship does hold.

   (b) Find a pair of positive real numbers x and y that demonstrates that $\sqrt{x \cdot y}$ need not equal $\sqrt{x} + \sqrt{y}$. Find a pair of non-zero values for which, by coincidence, this relationship does hold.

   (c) Is there a pair of non-zero real numbers x and y for which $2(x \cdot y) = 2x + 2y$? If so, describe how to find all such pairs.

2. A classmate claims that if everyone collectively forgot how to multiply two numbers together, logarithms would be useful for overcoming the memory lapse. That is, logarithms would be useful for computing products. Describe how you might still be able to compute the product $2 \cdot 3$ using the table of values for the logarithm function, given below, and the identity $\log(x \cdot y) = \log(x) + \log(y)$.

x	1	2	3	4	5	6	7	8	9
$\log(x)$	0	.301	.477	.602	.699	.778	.845	.903	.954

3. Another classmate claims that since 72 is the product of 8 and 9, the table can be used to calculate $\log(72)$. They claim that this is possible even though 72 is not an entry in the first row of the table. Moreover, they claim that the table can be used to compute the logarithm of any number which is the product of numbers represented in the first row of the table. Use the table and the identity $\log(x \cdot y) = \log(x) + \log(y)$ to:

   (a) Calculate $\log(15)$.

   (b) Calculate $\log(24)$ in two different ways.

   (c) Estimate $\log(17)$. Why isn't it possible to calculate $\log(17)$ precisely?

NAME: **CLASS ACTIVITY:** LOGARITHMS AND ISOMORPHISMS (page 1 of 4)

The steps used by astronomers and other scientists to calculate the product $x \cdot y$ using a logarithm function looked like the following:

- First, calculate $\log(x)$ and $\log(y)$.
- Next, add these values together.
- Finally, find a positive real number whose logarithm is the sum $\log(x) + \log(y)$. This number is $x \cdot y$.

For the first and last step, charts called log tables were created which recorded, as efficiently as possible, $\log(x)$ for all real numbers x. A (simplified) log table is included below.

	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	n/a	-1.000	-0.699	-0.523	-0.398	-0.301	-0.222	-0.155	-0.097	-0.046
1	0.000	0.041	0.079	0.114	0.146	0.176	0.204	0.230	0.255	0.279
2	0.301	0.322	0.342	0.362	0.380	0.398	0.415	0.431	0.447	0.462
3	0.477	0.491	0.505	0.519	0.531	0.544	0.556	0.568	0.580	0.591
4	0.602	0.613	0.623	0.633	0.643	0.653	0.663	0.672	0.681	0.690
5	0.699	0.708	0.716	0.724	0.732	0.740	0.748	0.756	0.763	0.771
6	0.778	0.785	0.792	0.799	0.806	0.813	0.820	0.826	0.833	0.839
7	0.845	0.851	0.857	0.863	0.869	0.875	0.881	0.886	0.892	0.898
8	0.903	0.908	0.914	0.919	0.924	0.929	0.934	0.940	0.944	0.949
9	0.954	0.959	0.964	0.968	0.973	0.978	0.982	0.987	0.991	0.996

For example, $\log(1.3)$ is found at the intersection of the row marked "1" and the column marked ".3", so $\log(1.3) = 0.114$. Log tables (and a related tool, the slide rule) were in use well into the 1900s, and were even used by NASA to make calculations for the Apollo 11 moon landing.

1. Use the above process and the log table provided to find the product of the following numbers:

   (a) 1.5 and 2.0

   (b) 0.5 and 8.6

   (c) 1.2 and 1.2

2. Another setting in advanced mathematics where a function "preserves" a difficult operation is found in linear algebra. Consider the set $GL_2(\mathbb{R}) = \{ \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] : a, b, c, d \in \mathbb{R}, ad - bc \neq 0\}$, which consists of all invertible 2×2 matrices with real number entries, and the determinant function $det : GL_2(\mathbb{R}) \to \mathbb{R}$, which assigns to each matrix its determinant.

   Recall from linear algebra that for any two matrices M and N in $GL_2(\mathbb{R})$, we have that $det(\mathrm{MN}) = det(\mathrm{M}) \cdot det(\mathrm{N})$. Choose two 2×2 matrices that you know are invertible and verify this identity holds for your two matrices.

3. Finn hopes that the determinant function (see previous question) can help them to avoid multiplying matrices, much like the logarithm helped astronomers avoid multiplication of large, real numbers. Finn models the following strategy based on the process given before Problem 1:

   - First, calculate $det(\mathrm{M})$ and $det(\mathrm{N})$
   - Next, multiply these values together.
   - Finally, find a matrix whose determinant is the product $det(\mathrm{M}) \cdot det(\mathrm{N})$. This matrix is MN.

   (a) Why won't Finn's strategy work?

   (b) What question might you ask Finn to help them see the flaw in this plan? Why do you think this question would be helpful?

4. What is a key property that the logarithm function applied to positive real numbers possesses that the determinant function applied to invertible matrices does NOT possess?

5. Between which two groups is the logarithm function an isomorphism?

6. Now that Finn understands that the function det is not a group isomorphism and will not help avoid matrix multiplication, they want to try and find another function from $(GL_2(\mathbb{R}), \cdot) \to (\mathbb{R}^*, \cdot)$ which is a group isomorphism, where $\mathbb{R}^*$ denotes the set of nonzero real numbers. Does such a function exist? Why or why not?

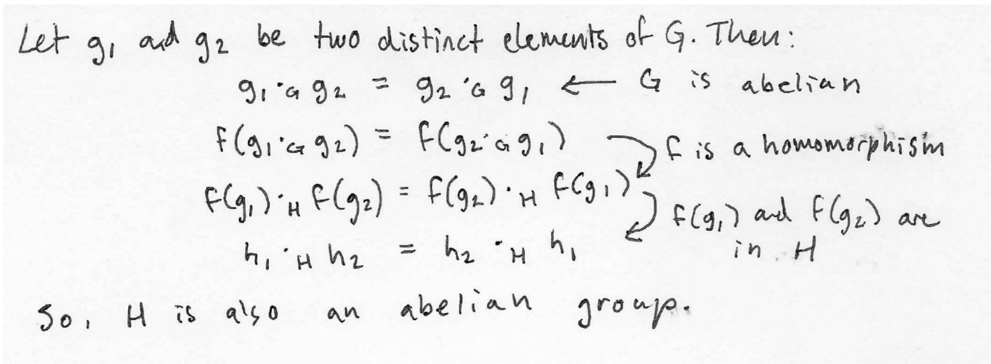
7. Finn shifts focus to finding a "better logarithm"; that is, a group isomorphism from $(\mathbb{R}^+, \cdot) \to (\mathbb{Z}, +)$ that allows them to multiply positive real numbers by adding integers instead of real numbers. Does such a function exist? Why or why not?

8. Maybe, Finn says, we could at least find a group isomorphism from $(\mathbb{Q}^+, \cdot) \to (\mathbb{Z}, +)$ that lets us multiply positive rational numbers by adding integers. Does such a function exist? Why or why not?

NAME: **HOMEWORK PROBLEMS:** LOGARITHMS AND ISOMORPHISMS (page 1 of 2)

1. Let $(G, \cdot_G)$ and $(H, \cdot_H)$ be groups and $f : G \to H$ be a homomorphism.
   (a) Show that the image of the group identity of G under f must be the group identity of H (that is, $f(e_G) = e_H$).
   (b) Use this result to show that, given an arbitrary element a in G, the image of the inverse of a under f must be the inverse of the image of a under f (that is, $\forall a \in G$, $f(a^{-1}) = f(a)^{-1}$).
   (c) Use the previous two results to show that the image of G under f is itself a group with the operation of H.

2. Let $(G, \cdot_G)$ and $(H, \cdot_H)$ be groups and $f : G \to H$ be a homomorphism. Furthermore, let G be an abelian group. Anjali writes the following proof, which she claims shows that H is also an abelian group:


Let g_1 and g_2 be two distinct elements of G. Then:

$g_1 \cdot_G g_2 = g_2 \cdot_G g_1$ ← G is abelian

$f(g_1 \cdot_G g_2) = f(g_2 \cdot_G g_1)$ ⟩ f is a homomorphism

$f(g_1) \cdot_H f(g_2) = f(g_2) \cdot_H f(g_1)$ ⟩ $f(g_1)$ and $f(g_2)$ are in H

$h_1 \cdot_H h_2 = h_2 \cdot_H h_1$

So, H is also an abelian group.

   (a) Explain why Anjali's proof does not show that H is an abelian group. What has she proven instead?
   (b) Given the same groups $(G, \cdot_G)$ and $(H, \cdot_H)$ let $f : G \to H$ now be an **isomorphism**. Add to Anjali's proof to show that, under these conditions, H is now an abelian group.

3. Imagine that you are a high school mathematics teacher who has noticed that one of your students, Hai, has assumed that $\log(x + y) = \log(x) + \log(y)$ when simplifying logarithmic expressions on a homework assignment.
   (a) Find a pair of real numbers x and y demonstrating that Hai's assumption is not always true. For which pairs of positive real numbers (x, y) is Hai's assumption true?
   (b) Using what you have learned about logarithms in this lesson, how might you help Hai understand that $\log(x \cdot y) = \log(x) + \log(y)$ is the correct identity? Why is your explanation helpful? Make sure your explanation is appropriate for a high school student.

4. Recall that $\mathbb{Z}_n$ is the set of *equivalence classes* on the integers, where two integers are in the same equivalence class if and only if they both have the same (smallest, non-negative) remainder when divided by n. The set $\mathbb{Z}_n$ contains n such equivalence classes which, canonically, are represented by the possible remainders when an integer is divided by n: $\{0, 1, \dots, n-2, n-1\}$.
   (a) Let θ be the map from $(\mathbb{Z}, +)$ to $(\mathbb{Z}_n, +)$ given by $\theta(z) = r$, where z is an integer and r is its remainder when divided by n. Show that θ is a group homomorphism. You may assume without proof that $(\mathbb{Z}, +)$ and $(\mathbb{Z}_n, +)$ are in fact groups.
   (b) Explain why θ cannot be an isomorphism in two ways: by showing that θ is not a bijection and by finding a difference in the algebraic structures of $(\mathbb{Z}, +)$ and $(\mathbb{Z}_n, +)$ (other than that their orders differ).

5. Let F be the set of continuous, real-valued functions on the interval $[0, 1]$. Let σ be the map from $(F, +)$ to $(\mathbb{R}, +)$ given by $\sigma(f) = \int_0^1 f(x)dx$ for all f in F. The operation "+" on F is defined by $(f + g)(x) = f(x) + g(x)$ for all f, g in F and $x \in [0, 1]$. The operation "+" on $\mathbb{R}$ is the usual real number addition.

HOMEWORK PROBLEMS: LOGARITHMS AND ISOMORPHISMS (page 2 of 2)

(a) Show that σ is a group homomorphism. You may assume without proof that $(F, +)$ and $(\mathbb{R}, +)$ are in fact groups.

(b) Describe the function that represents the identity element in $(F, +)$. Now, choose a different element of $(F, +)$ which maps to the identity element of $(\mathbb{R}, +)$ under the map σ. Draw a graph of your chosen function and explain how you know it meets this criteria.

(c) Let $ker(\sigma)$ be the set of **all** elements of $(F, +)$ that map to the identity element of $(\mathbb{R}, +)$ under the map σ. Show that $ker(\sigma)$ is a subgroup of $(F, +)$.

NAME: **ASSESSMENT PROBLEMS:** LOGARITHMS AND ISOMORPHISMS (page 1 of 2)

1. Let $M_2(\mathbb{R}) = \{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] : a, b, c, d \in \mathbb{R}\}$ and define the function $tr : (M_2(\mathbb{R}), +) \to (\mathbb{R}, +)$ to be the map $tr(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]) = a + d$. That is, tr assigns to each matrix its trace.

   (a) Show that tr is a group homomorphism. You may assume without proof that $(M_2(\mathbb{R}), +)$ and $(\mathbb{R}, +)$ are in fact groups.

   (b) Is tr an isomorphism? Why or why not?

2. Abina claims that there must be an isomorphism between $(\mathbb{Z}_4, +)$ and $(\mathbb{Z}_2 \times \mathbb{Z}_2, +)$ since the orders of the groups are the same.

   (a) Based on her claim, what do you think Abina understands about isomorphisms?

   (b) Explain why the two groups in question cannot be isomorphic.

   (c) What question would you ask Abina to help her understand her mistake? Why do you think your question would be helpful?