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Newton's Method

Single Variable Calculus

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1.1 Overview and Outline of Lesson

In high school and undergraduate mathematics classes, students often solve equations of the form $f(x) = 0$, where $f(x)$ is a polynomial function. When $f(x)$ is a quadratic function, finding the zeroes of f is relatively straightforward, because students can use the quadratic formula. When $f(x)$ is a polynomial function of higher degree, or any other nonlinear function, a way to estimate or explicitly find the zeroes (if possible) of the function is to use a linear approximation. Linearizing a nonlinear function is an often used technique across mathematics. This lesson emphasizes linear approximation as a useful technique for finding zeroes of a function, either by hand, or with technology, focusing on Newton's method as an example of an algorithmic way to numerically find the zeroes of a function. Undergraduates explore the geometry and calculus used to develop Newton's method, derive and apply the Newton's method procedure, and analyze hypothetical student work as an application to teaching. Because the study of linear functions is a core component of high school algebra, studying Newton's method provides prospective teachers an opportunity to develop a deeper understanding of and an increased fluency with linear functions.

1. Launch—Pre-Activity
   Undergraduates complete the Pre-Activity prior to the lesson. The purpose of the Pre-Activity is to review writing equations of tangent lines and algebraic methods for finding zeroes of a function. Undergraduates will also consider a function, which will be used in the Class Activity, whose zeroes cannot be found algebraically.

2. Explore—Class Activity

   - *Problems 1–4*:
     Undergraduates follow three hypothetical students' reasoning to algebraically and graphically create tangent lines to determine estimates for a zero of a function by finding *x*-intercepts of the tangent lines. They apply this reasoning three times to eventually sketch the first three iterations of Newton's method. Then they discuss which iterations produce a better estimate of the zero and consider how linearization is at the core of this process. If you are teaching this lesson over two class sessions, this may be an appropriate place to end Day 1.

- *Problem 5*:
  Undergraduates generalize the graphical and algebraic processes that they used in Problems 1–4 to describe Newton's method. Undergraduates then discuss when they can stop the iterative process.
- *Discussion—Newton's Method*:
  After Problems 1–5, the instructor formally defines Newton's method according to the instructional material of their course; then, undergraduates determine how to use technology to more quickly compute several iterations of the algorithm.

3. Apply and Extend—Class Activity
   - *Problem 6*:
     Undergraduates first practice applying the Newton's method algorithm to approximate the zeroes of a function. Based on this experience, they describe how to tell if a choice of x_0 will be a "good" initial guess for finding a given zero of a function.
4. Closure—Wrap-up
   The instructor concludes the lesson by describing how Newton's method provides a mathematical algorithm for estimating zeroes of functions that can be used by calculation devices in producing estimates of the zeroes of a given function. Taking time to discuss that linear approximations can be used by a calculation device to produce solutions to an equation provides future teachers perspectives for helping their future students see that many of the "answers" to these types of problems are approximations or estimates of a possible exact solution. As calculation devices are used quite extensively in secondary school mathematics courses, this helps strengthen understandings that procedures similar to Newton's method provide the mathematical foundation and algorithms used by the calculation devices to produce answers within a given error tolerance.

1.2 Alignment with College Curriculum

Newton's method is a topic that naturally fits into an "Applications of Differentiation" section in single variable calculus courses. Undergraduates are asked to consider methods for solving for zeroes that they commonly use (e.g., factoring) and are engaged in thinking about what to do when those algebraic methods are not applicable. Exploring and explaining why Newton's method works offers undergraduates the opportunity to learn about how Newton's method is an application of linear approximation and that linearization in general is a commonly used technique in mathematics for analyzing nonlinear functions.

1.3 Links to School Mathematics

By studying Newton's method, prospective teachers will develop deeper understanding of how an algorithmic method can be used to approximate zeroes of functions when the algebraic methods they teach fail to work. The *Mathematical Education of Teachers II* (CBMS, 2012) recommends that prospective teachers both derive results that may have been taken for granted in high school and that prospective teachers become familiar with technology. (Note that most graphing calculators do not use Newton's method in their algorithm; they likely use a version of a QR-Algorithm. See, for example, `http://sections.maa.org/okar/papers/2010/lloyd.pdf`).

This lesson highlights the following:

- Connections between calculus concepts and mathematical algorithms for calculating zeroes of a function;
- The use of linear functions as a tool for analyzing non-linear functions.

This lesson addresses several mathematical knowledge and practice expectations included in high school standards documents, such as the Common Core State Standards for Mathematics (CCSSM, 2010). The lesson emphasizes an application in which high school students use their knowledge of linear functions in a flexible manner. They look for key features (e.g., x-intercepts) of the graph of a function and use symbolic expressions coupled with graphical representations and the use of technology to reach resolutions to the tasks (c.f. CCSS.MATH.CONTENT.HSF.IF.C.7). The lesson relies on looking for and expressing regularity in repeated reasoning to uncover Newton's method and provides opportunities to consider the reasoning of others as well as construct sound arguments to support conclusions (c.f. CCSS.MATH.CONTENT.HSF.LE.B5, CCSS.MATH.PRACTICE.MP3).

1.4 Lesson Preparation

Prerequisite Knowledge

Undergraduates should know how to:

- Find the zeroes of a polynomial function both algebraically (when possible) and graphically;
- Compute derivatives of polynomial functions;
- Write an equation of a tangent line.

Learning Objectives

In this lesson, undergraduates will encounter ideas about teaching mathematics, as described in Chapter 1 (see the five types of connections to teaching listed in Table 1.2). In particular, by the end of the lesson undergraduates will be able to:

- Apply Newton's method, both graphically and algebraically, to approximate zeroes of a function;
- Flexibly use the derivative of a function to determine the slope of the tangent line to the graph of a function at a given point;
- Explain how to use Newton's method to compute the zeroes of a function;
- Analyze hypothetical student work in order to derive the procedure for Newton's method;
- Evaluate questions one might ask a hypothetical student to guide their understanding of Newton's method.

Anticipated Length

One or two 50-minute class sessions, depending on the instructor's pacing choices.

Materials

The following materials are required for this lesson.

- Pre-Activity (assign as homework prior to Class Activity)
- Class Activity (print out Problems 1–4 and 5–6 to pass out separately)
- Computer (for instructor to display a dynamic sketch during the Class Activity)
- Homework Problems (assign at the end of the lesson)
- Assessment Problems (include on quiz or exam after the lesson)

All handouts for this lesson appear at the end of this lesson, and LaTeX files can be downloaded from INSERT URL HERE.

1.5 Instructor Notes and Lesson Annotations

Before the Lesson

Assign the Pre-Activity as homework to be completed in preparation for this lesson. We recommend that you collect this Pre-Activity the day before the lesson so that you can review undergraduates' responses before you begin the Class Activity. This will help you determine if you need to spend additional time reviewing the responses to the Pre-Activity with your undergraduates.

Pre-Activity Review (10 minutes)

Discuss undergraduates' responses to the Pre-Activity as needed. The point of this Pre-Activity is to refresh undergraduates' memory of (1) how to write equations of tangent lines in point-slope form; (2) how to find zeroes of a function algebraically; and (3) the idea that you can't algebraically find zeroes of some functions. The last problem of the Pre-Activity contains the situation that undergraduates will consider in the Class Activity.

Pre-Activity Problems 1, 2 & 3

1. Write an equation of **a line** with slope 3 that passes through the point $(2, 1)$ in point-slope form. Then, write an equation of this line in slope-intercept form.

   Solution:
   The equation of the line in point-slope form is $y - 1 = 3(x - 2)$. Converted to slope-intercept form, this is $y = 3x - 5$.

2. Write an equation of the **tangent line** to the graph of $f(x) = x^2 + 2$ at the point $(1, 3)$ in point-slope form. Then, write an equation of this tangent line in slope-intercept form.

   Solution:
   Since $f'(x) = 2x$, $f'(1) = 2(1) = 2$ is the slope of the tangent line. Then, the equation of the line in point-slope form $y - 3 = 2(x - 1)$. Converted to slope-intercept form, this is $y = 2x + 1$.

3. More with tangent lines.

   (a) For a given function f, describe how to find an equation of the tangent line to the graph of f at $x = a$.

   Solution:
   To find the equation of a line, we need a point and slope. Then, we can substitute these values into the point-slope form of a line. To find the slope, we need to take the derivative of the function at $x = a$ (i.e., we need to find $f'(a)$). To find a point on the line, we can use $(a, f(a))$. When we substitute these values into the point-slope form of a line, we get $y - f(a) = f'(a)(x - a)$.

   (b) Now, write an equation of the tangent line to the graph of f at $x = a$.

   Solution:
   Calling this function g, we have that $g(x) = f'(a)(x - a) + f(a)$ or $y = f'(a)(x - a) + f(a)$.

Commentary:
When reviewing undergraduates' work for Problems 1–3, consider the following points:

- Verify that undergraduates are writing equations in point-slope form for Problems 1–3.
- Although Problems 2 and 3 may serve as review of concepts addressed earlier in your course, ensure that undergraduates effectively use the derivative of a function at a point to determine the slope of the corresponding tangent line on the graph of the function.
- Undergraduate responses to 3(a) should be in the form of written sentences that describe how and why their process will correctly produce an equation for the tangent line.

To motivate the Class Activity, facilitate a class discussion on Problems 4 and 5. Questions to ask about Problem 4 to motivate discussion include:

- How did you find the zeroes of each of these functions?
- Do these techniques always work? When is one technique preferred over another?

Pre-Activity Problem 4

4. Find the zeroes of the following functions.

   (a) $f(x) = x^2 - 4$

   Solution:
   Undergraduates will likely use one of two approaches:

Difference of Squares:

$$\begin{aligned} x^2 - 4 &= 0 \\ (x-2)(x+2) &= 0 \\ x \pm 2 &= 0 \\ x &= \pm 2 \end{aligned}$$

Solving for x:

$$\begin{aligned} x^2 - 4 &= 0 \\ x^2 &= 4 \\ x &= \pm 2 \end{aligned}$$

(b) $g(x) = 3x^2 + 7x - 2$

Solution:
Using the quadratic formula on $3x^2 + 7x - 2 = 0$:

$$x = \frac{-7 \pm \sqrt{49 - 4(3)(-2)}}{2(3)} = \frac{-7 \pm \sqrt{73}}{6}$$

(c) $h(x) = x^3 + x^2 - 2x$

Solution:
By factoring:

$$\begin{aligned} x^3 + x^2 - 2x &= 0 \\ x(x^2 + x - 2) &= 0 \\ x(x-1)(x+2) &= 0 \\ x &= 0, 1, -2 \end{aligned}$$

Commentary:
We have found that most undergraduates will factor or use the quadratic formula to find the zeros of these functions. Make sure that undergraduates recognize that these techniques only apply to some functions. As appropriate for your class, you may also want to discuss *why* each method works. For instance,

- When we factor, we use the fact that a product of real numbers can only be zero if at least one of the factors is zero.
- The quadratic formula is derived from completing the square.

Finally, ask undergraduates how finding zeroes of a function has arisen in our study of calculus. Discuss when finding zeroes is used in calculus (e.g., for identifying critical points and points of inflection) and discuss the following connection to teaching:

Discuss This Connection to Teaching

High school algebra courses focus on different techniques for finding zeroes of a function; these commonly include algebraic techniques (e.g., factoring) and graphing strategies (e.g., using graphing calculators, Desmos). While learning to solve for zeroes of polynomial functions remains the focus of many lessons, undergraduates may not have had opportunities to think about the reasons for focusing on solving for zeroes or situations in which these skills and techniques provide insight when solving problems. Contextualize the focus in high school algebra on solving for zeroes of functions by pointing out to undergraduates that solving for zeroes arises in the context of calculus, for example, with optimization problems.

Before reviewing Problem 5, let undergraduates know that the function they are considering in this problem will be used throughout the Class Activity.

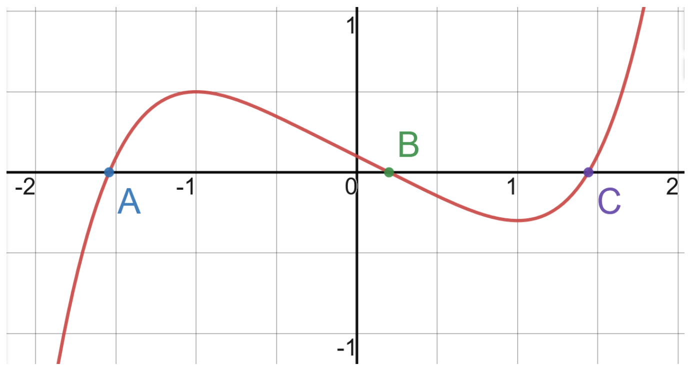
Pre-Activity Problem 5

5. Consider the function, $f(x) = \frac{1}{10}x^5 - \frac{1}{2}x + \frac{1}{10}$

   (a) Nnamdi has excellent algebra skills, yet he tries to find the zeroes algebraically and gets stumped. Explain why he is having trouble.

   Solution:
   $f(x)$ is a fifth degree polynomial. The usual algebraic methods of (e.g., solving explicitly for x, factoring, the quadratic formula) do not work for most quintic polynomials. In fact, there is no algebraic method that is guaranteed to solve quintic polynomials.

   (b) Nnamdi decides to graph f to find the zeroes. The zeroes are indicated on the graph as A, B, and C. Estimate the value of C.



   Solution:
   The answer rounded to the nearest thousandth is 1.441. From our experience, undergraduates will either make their best guess by looking at the graph or use a graphing utility to find a more exact answer.

   Commentary:
   The first part of the Class Activity focuses on the zero at C. Later, undergraduates will also examine the zeroes at A and B.

Ask undergraduates how a calculator or graphing utility might find the zero at C. Tell undergraduates that there is no general formula for algebraically finding zeroes of quintic polynomials explicitly and that, in general, for most non-linear functions the calculator is only able to estimate a zero very closely.

Discuss This Connection to Teaching

When they were in high school, undergraduates may have used calculators and computers to find the zeroes of a function. These tools use estimation methods that are hidden from the user. One goal of this lesson is to show undergraduates that these devices operate on programmed algorithms that use or extend mathematical concepts introduced in calculus. Furthermore, using calculus and their understanding of linear functions, they can develop an algorithm by hand for finding zeroes of a function and gain some insight into how their calculation device might be generating the zeroes. In addition to examining how calculation devices might be generating estimates, this lesson also provides undergraduates with a practical example of how linear functions are a useful tool when analyzing non-linear functions.

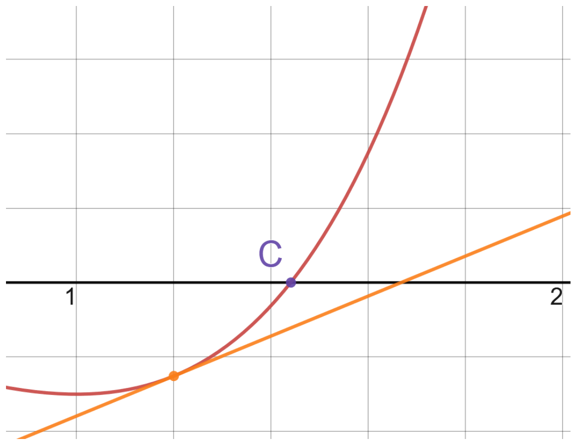
Class Activity: Problems 1–4 (30 minutes)

Pass out **Problems 1–4** of the Class Activity. Instruct undergraduates to work in small groups on Problem 1. See Chapter 1 for guidance on facilitating group work and selecting and sequencing student work for use in whole-class discussion.

Class Activity Problem 1

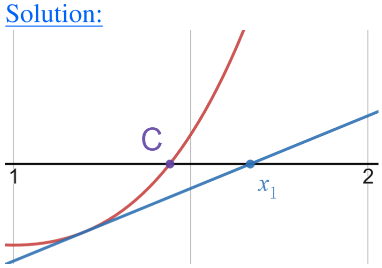
Recall the function from Problem 5 on the Pre-Activity, $f(x) = \frac{1}{10}x^5 - \frac{1}{2}x + \frac{1}{10}$, for which Nnamdi wanted to find the zeroes of the function. Nnamdi initially thinks that $x = 1.2$ is a good estimate of the zero, C, but when he zooms in on the graph he realizes that C is further to the right. He starts to experiment with linear functions to try to find a better estimate for C.

1. Nnamdi zooms in on the graph and sketches the tangent line at $x_0 = 1.2$ (see graph below).



(a) Label the x-intercept of Nnamdi's tangent line as x_1.

Solution:



(b) Write an equation of Nnamdi's tangent line in point-slope form and find the value of x_1.

Solution:
Since $f'(x) = \frac{1}{2}x^4 - \frac{1}{2}$, we can compute that $f'(1.2) \approx 0.537$. Furthermore, $f(1.2) \approx -0.251$. Taken together, we can write the equation of the tangent line as $y + 0.251 = 0.537(x - 1.2)$. Then, we can calculate the value of x_1 by finding the x-intercept of this line:

$$\begin{aligned} 0 + 0.251 &= 0.537(x_1 - 1.2) \\ 0.251 &= 0.537x_1 - 0.644 \\ 0.895 &= 0.537x_1 \\ x_1 &= 1.667 \end{aligned}$$

Commentary:
Circulate the room while undergraduates work. As you do so,

- Make sure undergraduates are using the algebraic methods from Problems 2 and 3 in the Pre-Activity.
- Watch for undergraduates who try to visually estimate the slope of (or a point on) the tangent line. Remind these undergraduates that they can find the slope by taking the derivative of the function at $x = 1.2$ and they can find a point on the tangent line by plugging $x = 1.2$ into the original equation.
- You may want to encourage undergraduates to round to the nearest thousandth because later questions will ask for accuracy to the thousandths place.

When you have noticed that most groups are done, bring the class back together for a whole group discussion. Briefly discuss undergraduates' responses to Problem 1(b) before instructing undergraduates to work in small groups on Problems 2–4.

Class Activity Problem 2

2. Taking inspiration from Nnamdi's idea, Mari decides to sketch another tangent line to the graph of $f(x)$ at the point $(x_1, f(x_1))$. She claims that the x-intercept of her tangent line will be closer to the zero C than x_1.

(a) Do you agree with Mari's claim? Explain why or why not.

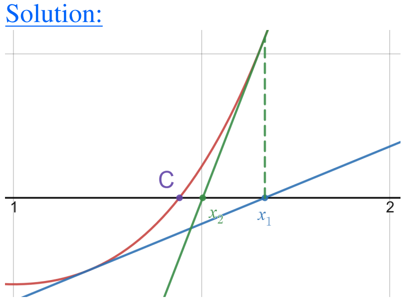
Sample Responses:

- Yes. It looks like the tangent line that Mari will draw is a better linear approximation of the function f, which will give a better approximation of the zero.
- No, since x_1 was farther away from the zero than Nnamdi's x_0.

Commentary:
We have found that most undergraduates will agree with Mari's claim. You might encourage undergraduates to consider whether Mari's claim would be true in general (i.e., for any choice of x_0 and any arbitrary function). Mari's claim is not always true. Undergraduates do not need to recognize this fact yet in their answer, however; throughout the lesson, they will see applications of Newton's method whose successive approximations do not always get closer to a zero.

(b) Sketch in Mari's tangent line. Label the x-intercept of her tangent line as x_2.

Solution:



Commentary:
As you circulate the room identify groups that could present their responses to the class (ideally, use a document camera to showcase their work). Alternatively, you can use undergraduates' work to reconstruct the tangent lines and zeroes using a computer graphing tool (e.g., Desmos) to share with the class.

(c) Write the equation of Mari's tangent line in point-slope form and find the value of x_2.

Solution:
Since $f'(x) = \frac{1}{2}x^4 - \frac{1}{2}$, we can compute that $f'(1.667) \approx 3.361$. Furthermore, $f(1.667) \approx 0.554$. Taken together, we can write the equation of the tangent line as $y - 0.554 = 3.361(x - 1.667)$. Then, we can calculate the value of x_2 by finding the x-intercept of this line:

$$\begin{aligned} 0 - 0.554 &= 3.361(x_2 - 1.667) \\ -0.554 &= 3.361x_2 - 5.603 \\ 5.049 &= 3.361x_2 \\ x_2 &= 1.502 \end{aligned}$$

Class Activity Problem 3

3. Amy uses both Mari's and Nnamdi's ideas to find a point, x_3, even closer to the zero C.

(a) What do you think she did? Explain.

Solution:
I think that Amy drew a tangent line to the graph of $f(x)$ at the point $(x_2, f(x_2))$. She labeled the x-intercept of this tangent line as x_3. Then, she found the equation of this tangent line and solved for its x-intercept, which is x_3.

Commentary:
If undergraduates struggle with this, ask them to describe Nnamdi's and Mari's procedures to you.

(b) Find the value of x_3.

Solution:
Since $f'(x) = \frac{1}{2}x^4 - \frac{1}{2}$, we can compute that $f'(1.502) \approx 2.045$. Furthermore, $f(1.502) \approx 0.113$. Taken together, we can write the equation of the tangent line as $y - 0.113 = 2.045(x - 1.502)$. Then, we can calculate the value of x_3 by finding the x-intercept of this line:

$$\begin{aligned} 0 - 0.113 &= 2.045(x_3 - 1.502) \\ -0.113 &= 2.045x_3 - 3.072 \\ 2.959 &= 2.045x_3 \\ x_3 &= 1.447 \end{aligned}$$

Commentary:

- As you circulate the room identify groups that could present their responses to the class (ideally, use a document camera to showcase their work). Alternatively, you can use undergraduates' work to reconstruct the tangent lines and zeroes using a computer graphing tool (e.g., Desmos) to share with the class.
- Encourage undergraduates to sketch a tangent line to $f(x)$ at $(x_2, f(x_2))$ and label the zero of the tangent line x_3. They should also (algebraically) find an equation for the tangent line to $f(x)$ at $(x_2, f(x_2))$ and determine the x-intercept to find x_3.

After a class discussion of Problems 2 and 3, review Nnamdi's, Mari's, and Amy's process. Then, ask your undergraduates why Nnamdi might have thought to use a line to begin with. From our experience, this question has prompted the following ideas from undergraduates:

- We can easily solve for the zero of a line (i.e., a first-degree polynomial function).
- Lines are "simpler" functions that, when zoomed in, approximate a continuous function.

Emphasize (or remind undergraduates) that linear approximations are generally much more accurate near the point of tangency, which is one reason that a good first guess will improve the likelihood that Newton's method works as expected. To supplement this idea, make sure to address the following connection to teaching during this discussion:

Discuss This Connection to Teaching

Linear functions (where "linear" refers to the fact that the graph of the function is a straight line) are an essential topic in a high school mathematics curriculum. Underscoring the importance of linear functions as a fundamental tool in simplifying complex problems in calculus and higher levels of mathematics provides prospective teachers insight into the many ways that linearization plays a critical role in mathematics and statistics. Moreover, they see that developing students' fluency with linear functions and related concepts will enhance students' capacity to use these ideas flexibly in future courses such as calculus.

We include a table in Problem 4 so that undergraduates can organize their work from Problems 1–3. The table helps undergraduates organize and collate their data in a manner that may better facilitate drawing conclusions based upon their calculations.

Class Activity Problem 4

4. Fill in the following table with the values of x_1, x_2, and x_3 that you found above. Describe what you notice about these values.

x_0	x_1	x_2	x_3
1.2			

Solution:
Filled in table:

x_0	x_1	x_2	x_3
1.2	1.667	1.502	1.447

Descriptions of what undergraduates notice will vary. Sample responses include:

- The first zero moves away from C, but afterwards the approximations move closer to C.
- From graphing, it seems like x_3 is very close to the real zero—it would be hard to draw the next iteration.

Commentary:
Undergraduates should be able to quickly fill in the table in Problem 4 using their work from Problems 1–3. You might give them a few minutes to observe patterns in the data in small groups before beginning a classroom discussion.

Advice on Delivering the Lesson Over Two Class Sessions

If you are teaching this lesson over two class sessions, this may be an appropriate place to end Day 1. See Chapter 1 for guidance on using **exit tickets** to facilitate instruction in a two-day lesson.

Class Activity: Problem 5 (15 minutes)

Pass out **Problems 5 and 6** of the Class Activity. Before instructing undergraduates to work in small groups on Problem 5, summarize the following connection to teaching:

Discuss This Connection to Teaching

For Problem 5, undergraduates will consider the details of Newton's method both graphically and algebraically. Using multiple representations is a key idea in high school mathematics. Links between algebraic and graphical representations of functions, for instance, are especially important in studying relationships and change.

As you circulate your classroom, you may want to identify groups to present their work to the class. (See chapter 1 for advice about selecting and sequencing student work for use in class.)

Class Activity Problem 5

5. The iterative process Amy follows from the work of Mari and Nnamdi is called Newton's method. To apply Newton's method, the process of "finding a tangent line at the point on the graph corresponding to the guess for the zero, finding its x-intercept, and using this x-intercept as the next guess for the zero" is repeated. These x-intercepts (usually denoted $x_0, x_1, x_2, x_3, \ldots$) provide successive approximations of the value of a zero of a function.

(a) Describe this process graphically.

Solution:

Make an initial estimate x_0 that is close to a zero of a given function. Sketch the tangent line to the curve at x_0. Then label the x-intercept of that tangent line. This x-intercept becomes x_1. Repeat this process of sketching the tangent line to the curve at x_1 and finding the x-intercept of the tangent line.

Commentary:

- Make sure undergraduates clearly identify the point on the function to which their line is tangent.
- After undergraduates share their ideas with the class, the dynamic sketch, found at `https://www.desmos.com/calculator/revqc4ybqz`, can help them visualize their ideas.

(b) Describe this process algebraically. Write out a formula to find x_{n+1}, the x-intercept of the tangent line created from the previous guess, x_n.

Solution:

1. Make an initial estimate x_0 that is close to a zero of a given function.
2. Write the equation of the tangent line,

$$y - f(x_n) = f'(x_n)(x - x_n)$$
$$y = f'(x_n)(x - x_n) + f(x_n)$$

3. Determine a new approximation by solving for the zero of this tangent line. That is, since x_{n+1} is the x-intercept of this line, we solve the equation $0 = f'(x_n)(x_{n+1} - x_n) + f(x_n)$ for x_{n+1}:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Commentary:

If undergraduates have difficulty getting started, ask them to reconsider their work in Problems 1–3 and think about the process they used. If further scaffolding is needed, help them notice that for the general process they should consider using $f(x)$ instead of the specific function used, say, in Problems 1–3.

(c) How do you know when to stop this iterative process? That is, when is your approximation of a zero "good enough?"

Sample Responses:

- You have to determine the appropriate accuracy. You repeat the process until the difference between iterations is smaller than the desired accuracy. For instance, if you need to be accurate to the thousandth, you need to repeat the process until $|x_{n+1} - x_n| < 0.001$.
- Referring to limitations of drawing on the graph: We are done after a couple of iterations.
- Never. You won't ever reach the zero.

Commentary:

- We have found that undergraduates tend to focus on the limitations of sketching successive lines on the graph or on the notion that somehow this process may never produce an exact zero. Emphasize the difference between closed form algebraic formulas for solving for zeroes versus approximations based upon iterative process such as this one.
- To further facilitate group interactions as undergraduates work on this problem, you might consider asking them to revisit Problem 4 and to then compute x_4 and x_5. If you choose to do this, consider the following questions:
  - Does the pattern you observed in Problem 4 hold? Do you see any new patterns?
  - What do you expect from the next iteration? Why?
  - What do you think it means to be "good enough" in this context?
  - When might we need a solution to be accurate to 3 decimal places? 8? 100?

Discussion: Newton's Method (5 minutes)

Formally define Newton's method according to the instructional material of your course. Undergraduates will calculate several iterations of Newton's method in Problem 6, so this may also be a good time to discuss or demonstrate ways in which technology can be used to ease calculations of successive Newton's method approximations.

Newton's method connects undergraduates' prior experiences with linear functions and their recent understandings about the relationship between the derivative of a function at a point and the slope of the tangent line to the graph of the function at that point. As undergraduates rediscover or develop Newton's method, they gain more experience with iterative processes in a context that builds upon these ideas. Their experiences in this lesson reinforce ideas about derivatives introduced early in the calculus course and rely on their having developed fluency with high school algebra concepts related to writing an equation of a line given a point and a slope and finding x-intercepts.

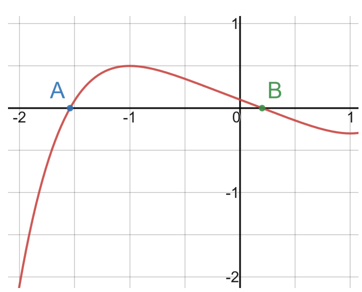
Class Activity: Problem 6 (15 minutes)

Instruct undergraduates to work in small groups on Problem 6. As a class, it may be helpful to first discuss 6(a)ii and compute the corresponding iterations for 6(b) if undergraduates seem to be unsure of how to proceed on this problem.

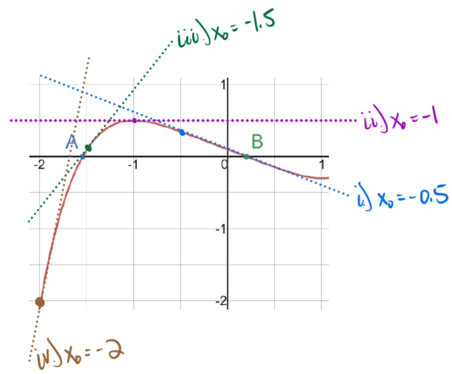
This problem focuses on the other two zeroes of $f(x) = \frac{1}{10}x^5 - \frac{1}{2}x + \frac{1}{10}$. The aim is to generate discussion about ways in which Newton's method can fail, how initial guesses can lead to different zeroes or none at all. Additionally, undergraduates engage in further practice in applying Newton's method.

Class Activity Problem 6

6. Reconsider $f(x) = \frac{1}{10}x^5 - \frac{1}{2}x + \frac{1}{10}$. Nnamdi now wants to use Newton's method to approximate the zero, A. He wonders what will happen if he uses the following initial guesses: $-0.5, -1, -1.5$, and -2.



For Problem 6(a), make sure undergraduates state what zero (A or B) they think each initial guess will lead to *before* they draw tangent lines on the graph. *After* drawing tangent lines, they should also note what zero (if any) they found. Below graphically shows (with tangent lines) what happens when you apply Newton's method using these initial guesses.



Class Activity Problem 6 : Part a

(a) Without doing any calculations, which zero of f do you expect each of these initial guesses to lead? Explain your reasoning. Use the graph above to graphically show (by drawing tangent lines) what happens when you apply Newton's method using these initial guesses.

i. $x_0 = -0.5$

Solution:

This guess leads to B. Sample explanation: The function f is very straight between $x = -0.5$ and the zero B, so the tangent line at $(-0.5, f(-0.5))$ will be a good linear approximation of the function and will cross the x-axis very close to B.

ii. $x_0 = -1$

Solution:

This guess does not lead to zero. The tangent line at $x = -1$ is horizontal and won't cross the x-axis.

Commentary:

It will help with the homework problems if undergraduates recognize that there is a horizontal tangent line at $x = -1$.

iii. $x_0 = -1.5$

Solution:

This guess leads to A. Sample explanation: -1.5 is already very close to the zero at A, so Newton's method will approximate that zero.

iv. $x_0 = -2$

Solution:

This guess leads to A. Sample explanation: Because of the shape of the graph around $x = -2$, the tangent lines won't ever cross the x-axis near any of the other zeroes.

In Problem 6(b), we recommend that undergraduates use technology to quickly apply Newton's method to find the zeros. Problem 6(c) provides an opportunity to discuss connections between the graphical and algebraic representations of Newton's method seen in 6(a) and 6(b).

Class Activity Problem 6 : Parts b & c

(b) Use Newton's method with all four initial guesses to calculate a zero of f. Give your answer to three decimal places, when applicable.

Solution:

i. $x_0 = -0.5 \to x_1 = 0.24 \to x_2 = 0.200$
ii. $x_1 = -1 \to$ No. There's a horizontal tangent line at x_1.
iii. $x_0 = -1.5 \to x_1 = 1.545 \to x_2 = -1.542$
iv. $x_0 = -2 \to x_1 = -1.72 \to x_2 = -1.579 \to x_3 = -1.544 \to x_4 = -1.542$

Commentary:
For each initial guess, ask undergraduates how many iterations it took to get a sufficiently accurate estimate of the zero. Compare the number of iterations required to get a "close enough" approximation when the initial guesses led to the same zero (i.e., for $x_0 = -1.5$ and $x_0 = -2$).

(c) Summarize to Nnamdi what you observe in the graph of f that indicates what zero you will approximate given your initial guess.

Solution:
Answers will vary. Key points that you might look for in undergraduate responses include:

- Choosing an initial guess very close to the desired zero is most effective for approximating that zero (compare speed of convergence for $x_0 = -1.5$ versus $x_0 = -2$).
- Choosing an initial guess that creates a horizontal tangent line causes Newton's method to fail (see: $x_0 = -1$).
- Choosing an initial guess that produces a tangent line that is a good linear approximation of the function near the zero is very effective for approximating that zero (see: $x_0 = -0.5$)

Commentary:
To discuss connections between the graphical and algebraic representations of Newton's method, consider asking the class the following prompts:

- Do all initial guesses lead to the same zero? Why or why not?
- Which initial guesses did (or did not) lead to the same zero? When two or more guesses led to the same zero, which of them reached that zero faster? Why do you think this is?
- We saw that choosing an initial guess which is also local extremum causes Newton's method to fail. How can you explain this failure graphically? Algebraically?

Wrap-Up (5 minutes)

Discuss what undergraduates explored throughout the Class Activity and how it relates to other ideas in calculus:

- During this lesson we applied Newton's method to estimate zeroes of a function when algebraic techniques are insufficient.
- Newton's method is an application of the idea of linear approximation. In order to work with tangent lines in this way, we must be able to find their slopes by taking a derivative.
- The iterative process and algorithm associated with Newton's method can be easily programmed into a programmable computing device.

Homework Problems

At the end of the lesson, assign the following homework problems.

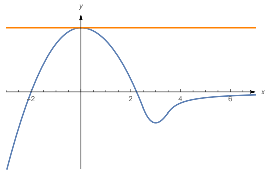
In the Class Activity, undergraduates saw that horizontal tangent lines lead to Newton's method failing to reach a zero. In Problem 1, undergraduates explore other ways that Newton's method can fail to converge. Having undergraduates formulate ideas about productive initial guesses versus unproductive ones engages undergraduates in the practice of using appropriate tools strategically and constructing viable reasons for their conclusions.

Homework Problem 1

1. The graph of $y = f(x)$ is shown here. Use the initial guesses given to determine which successfully lead to an approximation of a zero of the function f when using Newton's method. For each initial guess, graphically (by drawing tangent lines) support your conclusion based upon using Newton's method and explain your reasoning.

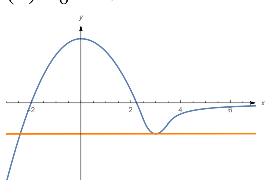
Solutions:

(a) $x_0 = 0$



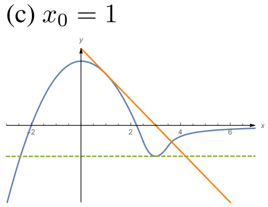
If $x_0 = 0$, the tangent line is horizontal. Thus, Newton's method fails.

(b) $x_0 = 3$



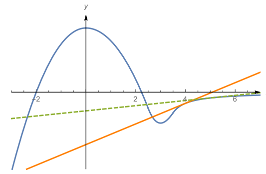
If $x_0 = 3$, the tangent line is horizontal. Thus, Newton's method fails.

(c) $x_0 = 1$



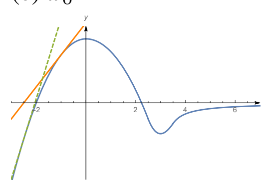
If $x_0 = 1$, then $x_1 = 3$ and we are now in the same situation as part (b). Thus, Newton's method fails.

(d) $x_0 = 4$



The iterations increase without bound. The sequence of iterations does not converge to a zero of the function. Thus, Newton's method fails.

(e) $x_0 = -1$

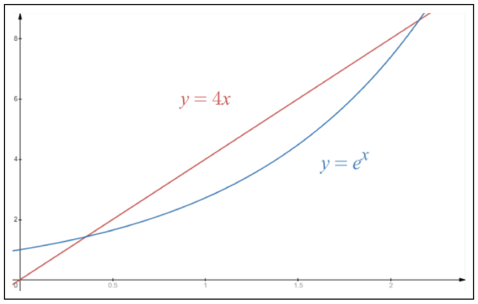


With this initial guess, the sequence of approximations will converge to -2. Thus, Newton's method works!

Problem 2 connects Newton's method to another area of secondary school mathematics: solving systems of equations. Just as we sometimes cannot solve for zeroes explicitly, we might also not be able to solve systems of equations using analytical methods. Instead, Newton's method gives us a way to find the solutions.

Homework Problem 2

2. Consider the system of equations given below.



$$y = e^x$$
$$y = 4x$$

(a) Explain how you could use Newton's method to approximate the two solutions to the system of equations.

Solution:
We create the function $h(x) = e^x - 4x$. The roots of this function are where $h(x) = 0$; that is, where $e^x - 4x = 0 \Rightarrow e^x = 4x$. Thus, when we apply Newton's method to find the roots of $h(x)$, we are also finding the solutions to the system.

(b) Choose any initial guess and calculate one iteration of Newton's method for each solution. Record your approximations up to six decimal places.

Sample Response:
First, note that $h'(x) = e^x - 4$. Using an initial guess of $x_0 = 0.5$ for the left-hand solution:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.5 - \frac{-0.351279}{-2.351279} \approx 0.350601$$

Using an initial guess of $x_0 = 2$ for the right-hand solution:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{-0.610943}{3.389056} \approx 2.180270$$

The next problem provides another example of Newton's method failing even though the tangent line is not horizontal—but this time, the reason for the failure is only obvious when explored algebraically rather than graphically. In Problem 3, undergraduates examine a hypothetical student's work and analyze questions one might ask the student to help guide them towards understanding how to adjust their work.

Homework Problem 3

3. Madalena is trying to use Newton's method to find the zeroes of the function $f(x) = x^3 - 2x + 2$. After making sure that $f'(1) \neq 0$, she chooses $x_0 = 1$. Then, she computes the first few iterations of Newton's method and begins a table of values:

x_0	x_1	x_2	x_3
1	0	1	

Madalena sees that this pattern will continue and comes to you for help.

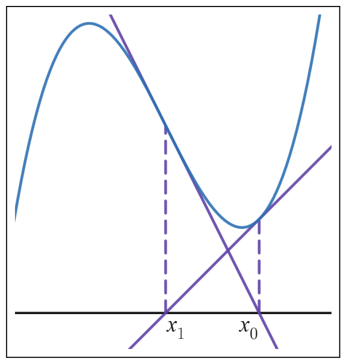
(a) Show that Madalena's computations for x_1 and x_2 are correct.

Sample Response:
With $f(x) = x^3 - 2x + 2$, $f'(x) = 3x^2 - 2$. Using Madalena's initial guess of $x_0 = 1$:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{1}{1} = 0$$
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0 - \frac{2}{-2} = 1$$

(b) Create a graph of f and sketch tangent lines to explain why Newton's method has failed.

Solution:
We see below that Newton's method fails because the x-intercept of the tangent line to the graph at x_1 is, again, x_0; i.e., $x_2 = x_0 = 1$. But because we use the same iterative process at each step, this next tangent line will take us back to x_1 again; i.e., $x_1 = x_3 = 0$. This will continue indefinitely: $x_{2n} = 1$ and $x_{2n+1} = 0$ for all non-negative integers n.



(c) Consider the following questions that you might ask Madalena:

i. Explain how the question below might help you assess what Madalena understands about Newton's method:

Given a graph of f and an initial guess x_0, how could you find x_1 without making the calculations you have already tried?

Sample Response:
It's not clear that Madalena knows what Newton's method looks like graphically. This question would help to assess whether or not Madalena understands what her calculations represent and how they (usually) produce closer approximations.

ii. Explain how following up with the next question might help Madalena to advance in her understanding of Newton's method:

Considering your graphical explanation of Newton's method, how could two iterations of Newton's method have the same value?

Sample Response:
This question would lead Madalena to the fact that the tangent line to the graph drawn at x_1 would have to "point back at" x_0. Then, she might understand that Newton's method has failed because it is stuck in a loop.

iii. Explain why the question below might not help Madalena:

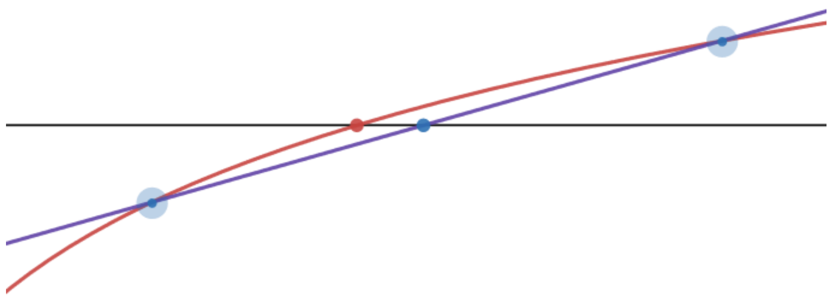
What is the formula for Newton's method?

Sample Response:
Madalena has already been using the correct formula to calculate Newton's method. Asking her to reproduce it wouldn't help her understand what's going on.

The purpose of Problem 4 is twofold: first, it allows undergraduates to connect Newton's method to the procedure you use to find the zero of a function with a graphing calculator (e.g., left bound, right bound, guess). This creates a connection to high school that may give prospective teachers a unique perspective on technology used in their classroom. Simultaneously, this question asks undergraduates to consider and respond to a reasonable (but flawed) suggestion from a hypothetical student.

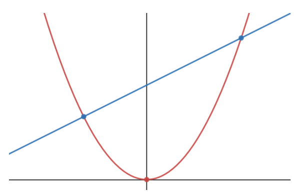
Homework Problem 4

4. Terrance, a high school algebra student, is using his TI-84 graphing calculator to find the zero of a function. To do so, the calculator requires him to choose a left bound (a point on the graph to the left of the zero), a right bound (a point on the graph to the right of the zero), and a guess (a point on the graph very close to the zero). Terrance thinks that the calculator is using the bounds he has chosen to construct a secant line, which it uses to approximate the zero. He draws the following example to illustrate his idea.



(a) Under what circumstances would Terrance's "secant method" fail to approximate a zero? Create a graph of one such example.

Solution:
This method will not work if the graph of the function touches the x-axis, but does not cross the x-axis (see graph below). In this case the secant line will not have a zero between the two bounds.



(b) Using your understanding of Newton's method (*but language that a high school algebra student would understand*), explain to Terrance why the calculator might need a left bound, right bound, and guess.

Sample Response:
When you give a calculator an initial guess, it uses that point to calculate an even closer guess afterwards; it does this by drawing a line that just touches the graph above your initial guess, then looking at where that line crosses the x-axis. The calculator does this many times to create a really good approximation, so the better your initial guess is, the faster and easier it is to get a really close estimate of the zero. Sometimes, even with a good guess, this process can lead to an unexpected zero. So, just to make sure that we don't accidentally find a different zero, the left and right bounds show the calculator where it should look.

Assessment Problems

The following two problems address ideas explored in the lesson, with a focus on connections to teaching and mathematical content. You can include these problems as part of your usual course quizzes or exams.

Assessment Problem 1

1. Use Newton's method with initial guess $x_0 = 3$ to calculate the first three approximations of a zero of the function $f(x) = x^2 - 5$. Be sure to use at least six decimal places.

   Solution:
   For $f(x) = x^2 - 5$ we know that $f'(x) = 2x$, and so

   $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3 - \frac{4}{6} = \frac{7}{3} \approx 2.333333$
   $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = \frac{7}{3} - \frac{4/9}{14/3} = \frac{47}{21} \approx 2.23810$
   $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = \frac{47}{21} - \frac{4/441}{84/21} = \frac{2207}{987} \approx 2.23607$

Assessment Problem 2

2. Jack and Raven are working together to find the zeroes of the function $f(x) = x^3 - 3x + 1$ using Newton's method. Jack suggests they begin with an initial guess of $x_0 = 1$. Raven says that won't work.

   (a) Why do you think Raven claims Jack's initial guess will not work? Using tangent lines, explain what Raven may have noticed.
   Solution:
   Raven sees that Jack's guess will not yield results because the tangent line at $x = 1$ is horizontal. Therefore, this tangent line does not cross the x-axis and Newton's method fails.

   (b) Write two questions Raven can ask Jack to help him revise his initial guess. Explain how Raven's questions might help Jack.
   Sample Responses:
   - Raven could ask Jack to draw the tangent line at $x = 1$. This would help Jack visualize that this tangent line does not cross the x-axis, which means they cannot proceed with Newton's method. Raven could then discuss how choosing an initial guess where a local maximum or local minimum occurs causes Newton's method to fail.
   - Raven could also ask Jack why he chose $x = 1$ as an initial guess and if there is a "better" initial guess. This might help Jack realize why $x = 1$ is an unproductive first guess and that he can choose an initial guess that is closer to one of the zeroes of the function.

1.6 References

[1] Conference Board of the Mathematical Sciences (2012). *The mathematical education of teachers II*. American Mathematical Society and Mathematical Association of America.

[2] National Governors Association Center for Best Practices & Council of Chief State School Officers (2010). *Common Core State Standards for Mathematics.* Authors. Retrieved from `http://www.corestandards.org/`

1.7 Lesson Handouts

Handouts for use during instruction are included on the pages that follow. LaTeX files for these handouts can be downloaded from INSERT URL HERE.

NAME: **PRE-ACTIVITY:** NEWTON'S METHOD (page 1 of 2)

1. Write an equation of **a line** with slope 3 that passes through the point $(2, 1)$ in point-slope form. Then, write an equation of this line in slope-intercept form.

2. Write an equation of the **tangent line** to the graph of $f(x) = x^2 + 2$ at the point $(1, 3)$ in point-slope form. Then, write an equation of this tangent line in slope-intercept form.

3. More with tangent lines.

   (a) For a given function f, describe how to find an equation of the tangent line to the graph of f at $x = a$.

   (b) Now, write an equation of the tangent line to the graph of f at $x = a$.

PRE-ACTIVITY: NEWTON'S METHOD (page 2 of 2)

4. Find the zeroes of the following functions.

   (a) $f(x) = x^2 - 4$

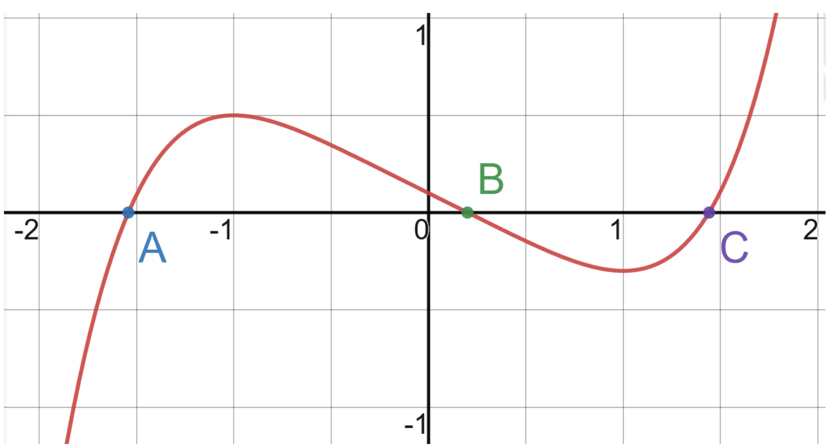
   (b) $g(x) = 3x^2 + 7x - 2$

   (c) $h(x) = x^3 + x^2 - 2x$

5. Consider the function, $f(x) = \frac{1}{10}x^5 - \frac{1}{2}x + \frac{1}{10}$

   (a) Nnamdi has excellent algebra skills, yet he tries to find the zeroes algebraically and gets stumped. Explain why he is having trouble.

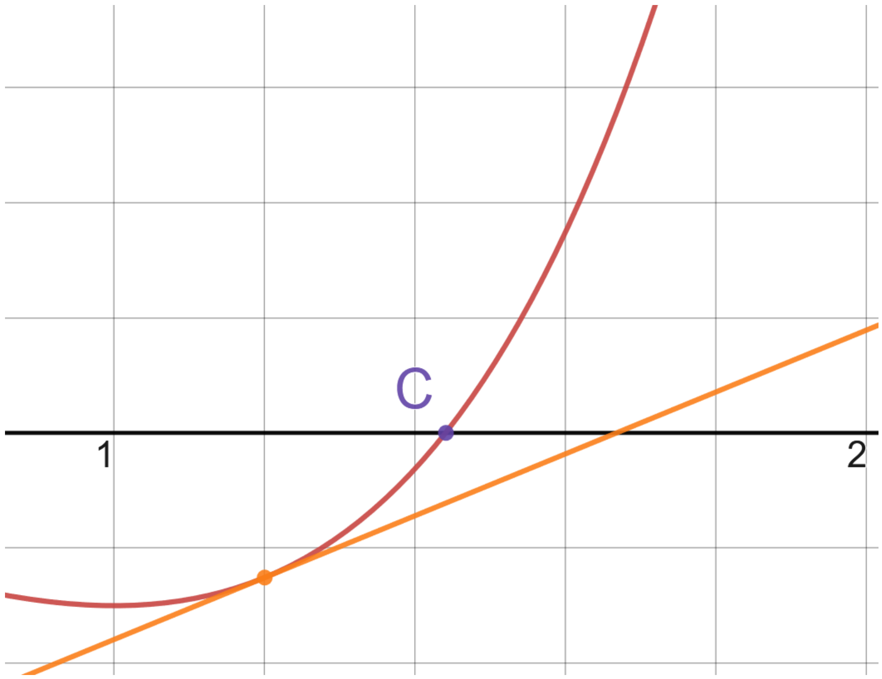
   (b) Nnamdi decides to graph f to find the zeroes. The zeroes are indicated on the graph as A, B, and C. Estimate the value of C.



NAME: **CLASS ACTIVITY:** NEWTON'S METHOD (page 1 of 4)

Recall the function from Problem 5 on the Pre-Activity, $f(x) = \frac{1}{10}x^5 - \frac{1}{2}x + \frac{1}{10}$, for which Nnamdi wanted to find the zeroes of the function. Nnamdi initially thinks that $x = 1.2$ is a good estimate of the zero, C, but when he zooms in on the graph he realizes that C is further to the right. He starts to experiment with linear functions to try to find a better estimate for C.

1. Nnamdi zooms in on the graph and sketches the tangent line at $x_0 = 1.2$ (see graph below).



   (a) Label the x-intercept of Nnamdi's tangent line as x_1.

   (b) Write an equation of Nnamdi's tangent line in point-slope form and find the value of x_1.

2. Taking inspiration from Nnamdi's idea, Mari decides to sketch another tangent line to the graph of $f(x)$ at the point $(x_1, f(x_1))$. She claims that the x-intercept of her tangent line will be closer to the zero C than x_1.

   (a) Do you agree with Mari's claim? Explain why or why not.

(b) Sketch in Mari's tangent line. Label the x-intercept of her tangent line as x_2.

(c) Write the equation of Mari's tangent line in point-slope form and find the value of x_2.

3. Amy uses both Mari's and Nnamdi's ideas to find a point, x_3, even closer to the zero C.

   (a) What do you think she did? Explain.

   (b) Find the value of x_3.

4. Fill in the following table with the values of x_1, x_2, and x_3 that you found above. Describe what you notice about these values.

x_0	x_1	x_2	x_3
1.2			

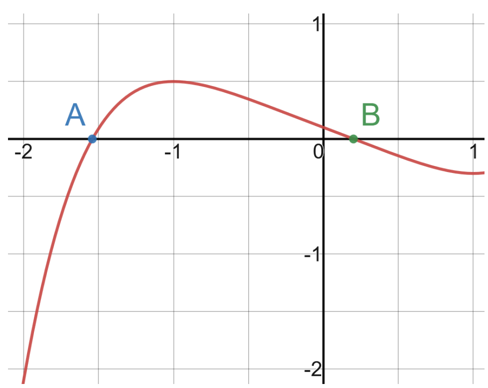
5. The iterative process Amy follows from the work of Mari and Nnamdi is called Newton's method. To apply Newton's method, the process of "finding a tangent line at the point on the graph corresponding to the guess for the zero, finding its x-intercept, and using this x-intercept as the next guess for the zero" is repeated. These x-intercepts (usually denoted x_0, x_1, x_2, x_3, $\ldots$) provide successive approximations of the value of a zero of a function.

   (a) Describe this process graphically.

   (b) Describe this process algebraically. Write out a formula to find x_{n+1}, the x-intercept of the tangent line created from the previous guess, x_n.

   (c) How do you know when to stop this iterative process? That is, when is your approximation of a zero "good enough?"

6. Reconsider $f(x) = \frac{1}{10}x^5 - \frac{1}{2}x + \frac{1}{10}$. Nnamdi now wants to use Newton's method to approximate the zero, A. He wonders what will happen if he uses the following initial guesses: -0.5, -1, -1.5, and -2.



(a) Without doing any calculations, which zero of f do you expect each of these initial guesses to lead? Explain your reasoning. Use the graph above to graphically show (by drawing tangent lines) what happens when you apply Newton's method using these initial guesses.

i. $x_0 = -0.5$

ii. $x_0 = -1$

iii. $x_0 = -1.5$

iv. $x_0 = -2$

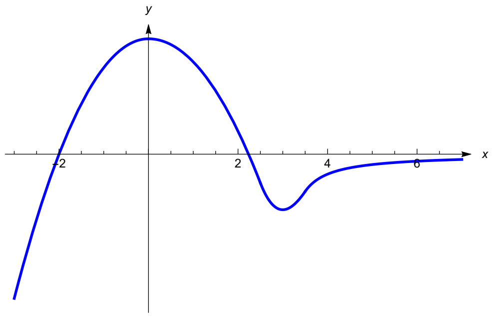
(b) Use Newton's method with all four initial guesses to calculate a zero of f. Give your answer to three decimal places, when applicable.

(c) Summarize to Nnamdi what you observe in the graph of f that indicates what zero you will approximate given your initial guess.

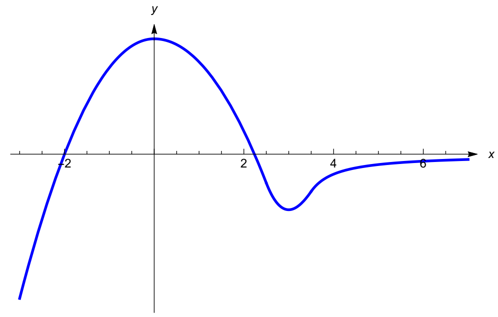
NAME: **HOMEWORK PROBLEMS:** NEWTON'S METHOD (page 1 of 3)

1. The graph of $y = f(x)$ is shown here. Use the initial guesses given to determine which successfully lead to an approximation of a zero of the function f when using Newton's method. For each initial guess, graphically (by drawing tangent lines) support your conclusion based upon using Newton's method and explain your reasoning.

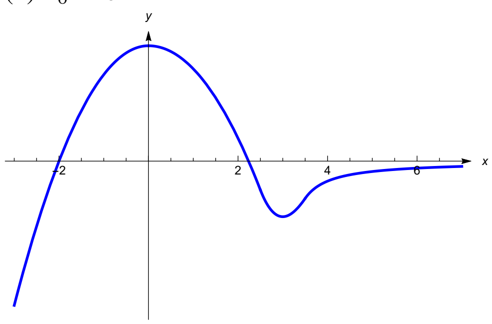
(a) $x_0 = 0$



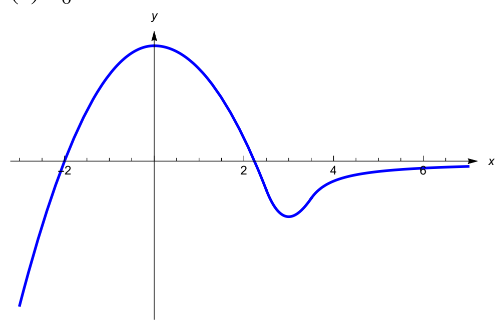
(d) $x_0 = 4$



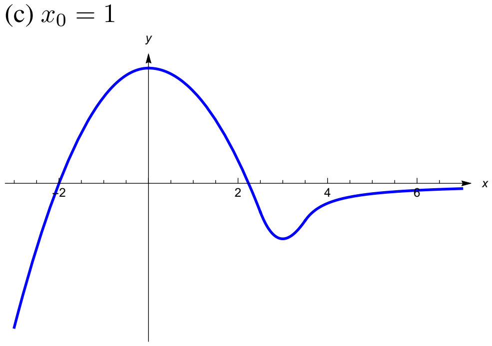
(b) $x_0 = 3$



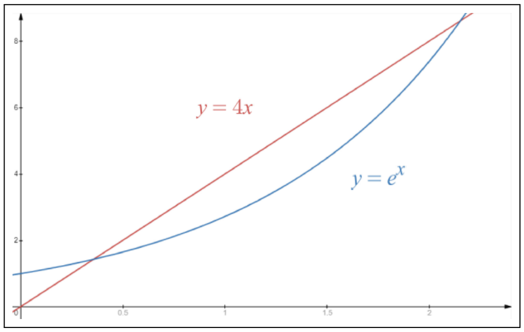
(e) $x_0 = -1$



(c) $x_0 = 1$



2. Consider the system of equations given below.



$$y = e^x$$
$$y = 4x$$

   (a) Explain how you could use Newton's method to approximate the two solutions to the system of equations.
   (b) Choose any initial guess and calculate one iteration of Newton's method for each solution. Record your approximations up to six decimal places.

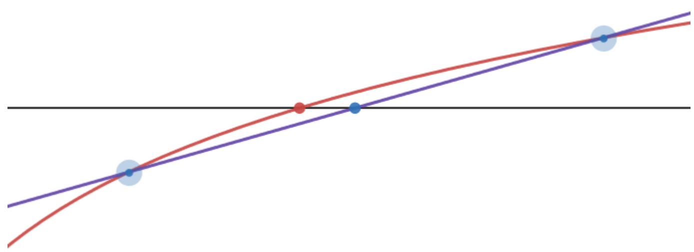
3. Madalena is trying to use Newton's method to find the zeroes of the function $f(x) = x^3 - 2x + 2$. After making sure that $f'(1) \neq 0$, she chooses $x_0 = 1$. Then, she computes the first few iterations of Newton's method and begins a table of values:

x_0	x_1	x_2	x_3
1	0	1	

   Madalena sees that this pattern will continue and comes to you for help.

   (a) Show that Madalena's computations for x_1 and x_2 are correct.
   (b) Create a graph of f and sketch tangent lines to explain why Newton's method has failed.
   (c) Consider the following questions that you might ask Madalena:
      i. Explain how the question below might help you assess what Madalena understands about Newton's method:
         Given a graph of f and an initial guess x_0, how could you find x_1 without making the calculations you have already tried?
      ii. Explain how following up with the next question might help Madalena to advance in her understanding of Newton's method:
         Considering your graphical explanation of Newton's method, how could two iterations of Newton's method have the same value?
      iii. Explain why the question below might not help Madalena:
         What is the formula for Newton's method?

4. Terrance, a high school algebra student, is using his TI-84 graphing calculator to find the zero of a function. To do so, the calculator requires him to choose a left bound (a point on the graph to the left of the zero), a right bound (a point on the graph to the right of the zero), and a guess (a point on the graph very close to the zero). Terrance thinks that the calculator is using the bounds he has chosen to construct a secant line, which it uses to approximate the zero. He draws the following example to illustrate his idea.



   (a) Under what circumstances would Terrance's "secant method" fail to approximate a zero? Create a graph of one such example.

   (b) Using your understanding of Newton's method (*but language that a high school algebra student would understand*), explain to Terrance why the calculator might need a left bound, right bound, and guess.

NAME: **ASSESSMENT PROBLEMS:** NEWTON'S METHOD (page 1 of 1)

1. Use Newton's method with initial guess $x_0 = 3$ to calculate the first three approximations of a zero of the function $f(x) = x^2 - 5$. Be sure to use at least six decimal places.

2. Jack and Raven are working together to find the zeroes of the function $f(x) = x^3 - 3x + 1$ using Newton's method. Jack suggests they begin with an initial guess of $x_0 = 1$. Raven says that won't work.

   (a) Why do you think Raven claims Jack's initial guess will not work? Using tangent lines, explain what Raven may have noticed.

   (b) Write two questions Raven can ask Jack to help him revise his initial guess. Explain how Raven's questions might help Jack.