1

Solving Equations in Alternative Number Systems

Abstract (Modern) Algebra I

Andrew Kercher, *Simon Fraser University*
James A. M. Álvarez, *The University of Texas at Arlington*
Kyle Turner, *The University of Texas at Arlington*

1.1 Overview and Outline of Lesson

Many undergraduate courses in abstract algebra include a learning goal of developing the capacity for students to reason abstractly about mathematical structures. As such, this reasoning is often applied in the context of highlighting the mathematical structures that make familiar high school algebra techniques possible. However, attempting to apply these familiar techniques to solving polynomial equations over finite integer rings leads to outcomes that undergraduates might not expect. For example, a linear equation with finitely many solutions might still have more than one solution; similarly, quadratic equations may have more than two real roots. Ultimately, these observations can be leveraged to motivate undergraduates to think about the ring structure for which their familiar techniques work and eventually to determine for which n the ring $\mathbb{Z}_n$ is an integral domain. This lesson focuses on building intuition about the ring $\mathbb{Z}_n$ while also embedding targeted opportunities for reflecting on the reasoning behind various methods used to solve polynomial equations from high school mathematics.

1. Launch—Pre-Activity
   Prior to the lesson, undergraduates complete the Pre-Activity. First, undergraduates identify which elements of $\mathbb{Z}_{10}$ have the same algebraic behavior as certain rational numbers. Then, they graph the solution set of a linear equation in two variables in both $\mathbb{Z}_5 \times \mathbb{Z}_5$ and $\mathbb{Z}_6 \times \mathbb{Z}_6$. Finally, undergraduates describe the precise algebraic steps used to solve a linear equation of one variable in $\mathbb{R}$ and attempt to transfer these steps to $\mathbb{Z}_5$ and $\mathbb{Z}_6$, where they encounter difficulty if that ring is not also a field. Instructors can launch this lesson by reviewing the answers to the Pre-Activity.

2. Explore—Class Activity

   - *Problems 1–3*:
     Using their graphical representations from Problem 2 of the Pre-Activity, undergraduates are informally introduced to the idea of a zero divisor in $\mathbb{Z}_6$; this lays the groundwork for undergraduates' capacity to later deduce that no zero divisor can be a unit. In Problem 3, a hypothetical student makes a conjecture which undergraduates then evaluate. In doing so, they consider which elements of $\mathbb{Z}_n$ are zero divisors and which

are units. By comparing these lists, undergraduates deduce that any element of $\mathbb{Z}_n$ represented by an integer that is relatively prime with n is a unit but is not a zero divisor.

- *Discussion—The Structure of* $\mathbb{Z}_n$:
  The instructor leverages undergraduates' work on Problems 1–3 to motivate the definitions of zero divisors and integral domains. Then, they prove (or discuss) a number of results that culminate in the conclusion that $\mathbb{Z}_p$ is a field and integral domain when p is prime. Central to this claim is the fact that any element of $\mathbb{Z}_n$ which is relatively prime with n is a unit and not a zero divisor.
- *Problems 4–6*:
  Undergraduates examine two techniques for solving quadratic equations: factoring and the application of the standard quadratic formula. They find that both techniques can be problematic in $\mathbb{Z}_{10}$.

3. Closure—Wrap-Up
   Instructor wraps up the lesson by reiterating that $\mathbb{Z}_n$ is both a field and an integral domain when n is prime. In this case, many familiar algebraic techniques from secondary school can still be used to solve equations. When n is composite, the application of these techniques proves to be problematic.

1.2 Alignment with College Curriculum

A study of $\mathbb{Z}_n$ appears in undergraduate abstract algebra courses and in introductory number theory courses. It is the classic example of a finite commutative ring, and attempting to solve polynomial equations in $\mathbb{Z}_n$ leads naturally to definitions of zero divisors and units. This lesson builds a foundation from which lessons on rings of polynomials, factorization, and algebraic closure could be built.

1.3 Links to School Mathematics

Solving equations is a significant component of high school algebra. The majority of the experiences in equation-solving occur over $\mathbb{R}$, but the field properties of $\mathbb{R}$ that underlie the process of equation-solving are typically not emphasized. This lesson addresses the Common Core State Standards for Mathematics (CCSSM, 2010) related to solving linear equations and quadratic equations and attends to the properties of $\mathbb{R}$ "taken for granted" by exploring the same material in alternative settings. In particular, these properties include the zero product property and the existence of a multiplicative inverse for every nonzero element.

This lesson highlights:

- Solving equations in $\mathbb{Z}_n$ can be both similar to and different from solving equations in $\mathbb{R}$;
- Visualizing solutions to equations in $\mathbb{Z}_n$ can be both similar to and different from visualizing solutions in $\mathbb{R}$.

This lesson addresses several mathematical knowledge and practice expectations included in high school standards documents, such as the CCSSM. For example, high school students are regularly taught how to solve both linear and quadratic equations of one variable (c.f. CCSS.MATH.CONTENT.HSA.REI.B.3, CCSS.MATH.CONTENT.HSA.REI.B.4). In the case of a quadratic equation, there are multiple associated algebraic techniques of which high school students are expected to develop a comprehensive understanding (see CCSS.MATH.CONTENT.HSA.REI.B.4.B). One such technique is factoring, which exemplifies the types of algebraic manipulations that high school students are expected to master when working with equations—namely, those manipulations that do not change the expected solution set of the equation (see CCSS.MATH.CONTENT.HSA.SSE.B.3). Finally, this lesson and its activities aim to provide prospective teachers with opportunities to construct mathematical arguments, analyze and respond to the arguments of others, and to critique the underlying reasoning of such an argument.

1.4 Lesson Preparation

Prerequisite Knowledge

Undergraduates should know:

- The definition of a ring (i.e., the ring axioms);

- The definitions of unit and field;
- The definition of $\mathbb{Z}_n$ and how addition and multiplication are conducted in this ring.

Learning Objectives

In this lesson, undergraduates will encounter ideas about teaching mathematics, as described in Chapter 1 (see the five types of connections to teaching listed in Table 1.2). In particular, by the end of the lesson undergraduates will be able to:

- Justify when an element of $\mathbb{Z}_n$ is a zero divisor or a unit;
- Contrast the process of solving quadratic equations in $\mathbb{Z}_n$ with the process of solving in $\mathbb{R}$ by:
  - Explaining how factoring outside of an integral domain might fail to find the entire set of roots;
  - Explaining how the quadratic formula is more difficult to apply in general, but especially outside of a field.
- Analyze hypothetical student work or conjectures to explore student thinking about solving equations in $\mathbb{Z}_n$;
- Pose guiding questions to help a hypothetical student determine when a ring is an integral domain.

Anticipated Length

One 75-minute class session.

Materials

The following materials are required for this lesson.

- Pre-Activity (assign as homework prior to Class Activity)
- Class Activity
- Homework Problems (assign at the end of the lesson)
- Assessment Problems (include on quiz or exam after the lesson)

All handouts for this lesson appear at the end of this lesson, and LaTeX files can be downloaded from INSERT URL HERE.

1.5 Instructor Notes and Lesson Annotations

Before the Lesson

Assign the Pre-Activity as homework to be completed in preparation for this lesson.

> *We recommend that you collect this Pre-Activity the day before the lesson so that you can review undergraduates' responses before you begin the Class Activity. This will help you determine if you need to spend additional time reviewing the solutions to the Pre-Activity with your undergraduates.*

Pre-Activity Review (10 minutes)

Discuss the solutions to the Pre-Activity as needed; if you see that most undergraduates completed each problem correctly, you may not need to spend much time reviewing or discussing the solutions.

> *The Pre-Activity is designed to re-familiarize undergraduates with modular arithmetic in $\mathbb{Z}_n$ by asking them to make basic computations, visualize lines, and solve equations of one variable in several different finite integer rings. These are all skills that will be used directly in the Class Activity.*

Probe undergraduates' understanding of Problem 1(b) as it can be referenced later to facilitate Problem 3(c) of the Class Activity, if needed. Use questions similar to the following to generate additional discussion:

- What element of $\mathbb{Z}_{10}$ do you think might represent "$^3/_7$"? Is this the only such element? Explain.
- What element(s) of $\mathbb{Z}_{10}$ do you think might represent "$\sqrt{5}$"? Is this the only such element? Explain.
- Based on your answers to Problem 1(b), which are the units of $\mathbb{Z}_{10}$?

Pre-Activity Problem 1

1. Recall that $\mathbb{Z}_n$ is the set of *equivalence classes* on the integers, where two integers are in the same equivalence class if and only if they both have the same (smallest, non-negative) remainder when divided by n. The set $\mathbb{Z}_n$ contains n such equivalence classes which, canonically, are represented by the possible remainders when an integer is divided by n: $\{0, 1, \ldots, n-2, n-1\}$.

   (a) Fill in the following chart with the representative of each integer's equivalence class in $\mathbb{Z}_{10}$.

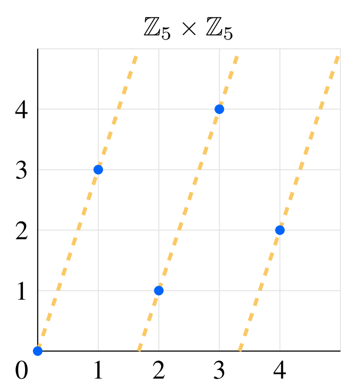
Integer	36	17	-4	-17
Representative in $\mathbb{Z}_{10}$	6	7	6	3

   If we are careful, we can also (sometimes) represent non-integers as elements of $\mathbb{Z}_n$. For example, if we interpret the notation "$^1/_3$" as "the element you multiply by 3 to get 1," we would then consider 7 in $\mathbb{Z}_{10}$ a representative of "$^1/_3$", since $3 \cdot 7 = 21 = 1$ (where $21 = 1$ because 21 has remainder 1 when divided by 10). Furthermore, no other element of $\mathbb{Z}_{10}$ has this property.

   (b) Fill in the following chart with the representative in $\mathbb{Z}_{10}$, if it exists.

"Non-integer"	"$^1/_1$"	"$^1/_2$"	"$^1/_3$"	"$^1/_4$"	"$^1/_5$"	"$^1/_6$"	"$^1/_7$"	"$^1/_8$"	"$^1/_9$"
Representative in $\mathbb{Z}_{10}$	1	X	7	X	X	X	3	X	9

In Problem 2, we found it useful to ask undergraduates to visualize the graph as a continuous geometric line with slope 3 which "wraps around" the finite space. To see this, we "extend" the grid to include the space "between" $n-1$ and n as shown here.



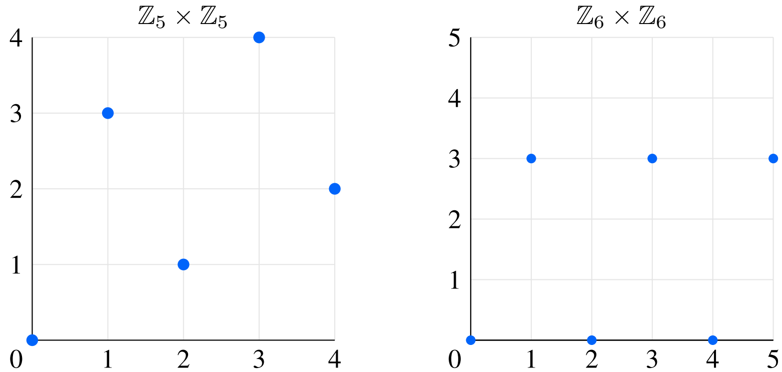
Undergraduates will be directed to refer back to and use their graphs when they answer Problems 1 and 2 of the Class Activity, so make sure that it is clear what is and is not part of the graph if discussing this representation.

Pre-Activity Problem 2

2. Let A be a set of elements (numbers) with a well-defined notion of addition and multiplication. We define a *line over A* as the solution set to an equation of the form $ax + by = c$ for some fixed values of $a, b, c \in$ A. That is, a line is the set of all ordered pairs $(x, y) \in$ A $\times$ A that make $ax + by = c$ a true statement in A. The *graph of a line* is a scatter plot of the solution set on a coordinate plane, usually one with perpendicular axes marked by the elements of A.

   For example, in the system of real numbers, the set of all ordered pairs that make $y = 3x$ a true statement in $\mathbb{R}$ has a graph which is a continuous straight geometric line of slope 3 through the point $(0, 0)$ in our usual Cartesian coordinate system.

Graph the line $y = 3x$ in the following spaces on the provided axes.



Facilitate a class discussion focused on why it would be useful to create a graph of an equation. Prompt undergraduates to consider both the pros and cons of visualizing abstract constructs—for example, you might touch on the fact that using a dotted line to produce the graph is useful but could also be misleading if it is incorrectly taken to be part of the graph itself. Discuss the following connection to teaching:

Discuss This Connection to Teaching

High school teachers are expected to give their students opportunities to work with both symbolic and visual representations of mathematical concepts. This context focuses prospective teachers' attention on the meaning of the linear equation that may be overlooked when considering equations over $\mathbb{R}$. By asking them to transition between symbolic and visual representations of a line in unconventional spaces, we model pedagogical practices that they will need to replicate in their future classrooms and look back to high school topics from a different point of view.

The questions in Problem 3 are intended to elicit undergraduate thinking about assertions of equality and whether usual algebraic manipulations are applicable and reliably produce the entire solution set. Encourage undergraduates to think in terms of ring structure by using questions such as the following:

- The ring axioms are defined using addition and multiplication; what do we mean if we talk about "subtraction" and "division" in a ring? Do these concepts always exist?
- How might we justify that "adding the same quantity to each side" preserves equality in rings other than $\mathbb{R}$?

Pre-Activity Problem 3

3. In solving the equation $x + 4 = 1 + 4x$ in $\mathbb{R}$, a student makes the following algebraic manipulations:

$$\begin{aligned} x + 4 &= 1 + 4x \\ 4 &= 1 + 3x \\ 3 &= 3x \\ 1 &= x \end{aligned}$$

The student then concludes that $x = 1$ is the solution set to $x + 4 = 1 + 4x$ in $\mathbb{R}$.

(a) Describe the mathematical justification for each step in the student's solution.

Solution:

The student is using additive and multiplicative inverses to simplify the equation until it's in a form where the solution is evident. First, the student adds $-x$ to each side; then, they add -1 to each side; finally, they multiply everything by $\frac{1}{3}$. The last line shows that the entire solution set is $\{1\}$.

(b) To solve $x + 4 = 1 + 4x$ for x in $\mathbb{Z}_5$, are we allowed to repeat the process the student used (in $\mathbb{R}$) as presented above? Does this process yield the entire solution set to the equation? Explain.

Solution:

We must make some adjustments, but the basic idea of each step is still sound in $\mathbb{Z}_5$:

- Because $-x = 4x$ in $\mathbb{Z}_5$, we first add $4x$ to each side: $x+4 = 1+4x \Rightarrow 5x+4 = 1+8x \Rightarrow 4 = 1 + 3x$.
- Because $-1 = 4$ in $\mathbb{Z}_5$, we next add 4 to each side: $4 = 1 + 3x \Rightarrow 8 = 5 + 3x \Rightarrow 3 = 3x$.
- Finally, because $3 \cdot 7 = 21 = 1$, $1/3 = 7$ in $\mathbb{Z}_5$. So we multiply both sides by 7, and: $3 = 3x \Rightarrow 21 = 21x \Rightarrow 1 = x$.

Just like in the real numbers, we see from here that the solution set is $\{1\}$.

(c) To solve $x + 4 = 1 + 4x$ for x in $\mathbb{Z}_6$, are we allowed to repeat the process the student used (in $\mathbb{R}$) as presented above? Does this process yield the entire solution set to the equation? Explain.

Solution:

If we attempt to make the same adjustments as in $\mathbb{Z}_5$:

- Now $-x = 5x$ in $\mathbb{Z}_6$, so we add $5x$ to each side: $x + 4 = 1 + 4x \Rightarrow 6x + 4 = 1 + 9x \Rightarrow 4 = 1 + 3x$.
- Similarly, because $-1 = 5$ in $\mathbb{Z}_6$, we next add 5 to each side: $4 = 1 + 3x \Rightarrow 9 = 6 + 3x \Rightarrow 3 = 3x$.
- At this point, however, we can check by exhaustion that none of the elements of $\mathbb{Z}_6$ function like $1/3$: that is, $\forall\, x \in \mathbb{Z}_6, 3x \neq 1$.

Because we cannot simplify any further, we must check by inspection to find that the solution set is $\{1, 3, 5\}$.

Commentary:

Make sure undergraduates take away the following points from this problem.

- "Division" may not be well-defined in $\mathbb{Z}_n$ if not all of its elements are units (i.e., if it is not a field).
- In reasonably small finite rings such as $\mathbb{Z}_5$ or $\mathbb{Z}_6$, it is easy to check for solutions to equations by exhaustion.

Wrap up the Pre-Activity by discussing and summarizing ways that finite integer rings do not have the familiar structure of $\mathbb{R}$ and, as a result, certain ideas we have about solving linear equations are not preserved. Even when written as simply as $ax = b$, the solution to a linear equation may not be as straightforward as we might have hoped.

Class Activity: Problems 1–3 (25 minutes)

To introduce the lesson, discuss the following idea with your class:

Solving an equation of one variable entails listing or representing the numbers which, when written in place of the variable, yield a true mathematical sentence. This means we may think of an equation as a way of stating a property, and the solution set of that equation as the collection of numbers that have that property. Typically, we identify the solutions by translating the equation into equivalent but simpler forms until the set of numbers that make the equation true is evident. In making these translations, we appeal to the properties of the number system within which the equation is being solved. As we saw in the Pre-Activity, we need to understand what properties the finite integer rings possess so that we can reliably use equations in this way.

Distribute the Class Activity. Instruct undergraduates to work on the Problems 1 and 2 in small groups. See Chapter 1 for guidance on facilitating group work and selecting and sequencing student work for use in whole-class discussion. It may be helpful to illustrate (or offer a hint about) how Problems 1 and 2 can be solved visually by drawing the graph of the line $y = 3x$ in $\mathbb{R} \times \mathbb{R}$. Ask undergraduates to think about how the change in domain affects the usual way we find all of the solutions to $0 = 3x$ in $\mathbb{R} \times \mathbb{R}$ by locating where the graph of the line crosses the x-axis (that is, where $y = 0$). Encourage them to use the graphs of the lines in $\mathbb{Z}_5 \times \mathbb{Z}_5$ and $\mathbb{Z}_6 \times \mathbb{Z}_6$ from the Pre-Activity for these problems.

Some of the following questions can be used to motivate discussion:

- How could we have solved these problems if we didn't already have the relevant graphs on hand?
- Since no solution exists to the equation $1 = 3x$ in $\mathbb{Z}_6$, what does that tell us about the element $3 \in \mathbb{Z}_6$?
- Why do you think $\mathbb{Z}_5$ appears to behave more like $\mathbb{R}$ than $\mathbb{Z}_6$ does?

Class Activity Problems 1 & 2

Consider the linear equation $y = 3x$ (and your corresponding graphs) from the Pre-Activity.

1. How many solutions to $0 = 3x$ exist in the following domains? What are they?

Domain	$\mathbb{R}$	$\mathbb{Z}_5$	$\mathbb{Z}_6$
# of Sol.	1	1	3
Sol. Set	$\{0\}$	$\{0\}$	$\{0, 2, 4\}$

2. How many solutions to $1 = 3x$ exist in the following domains? What are they?

Domain	$\mathbb{R}$	$\mathbb{Z}_5$	$\mathbb{Z}_6$
# of Sol.	1	1	0
Sol. Set	$\{\frac{1}{3}\}$	$\{2\}$	$\emptyset$

Before allowing undergraduates to move on to Problem 3, hold a brief whole-class discussion to discuss key answers to Problems 1 and 2 and also to verify that the class is well-positioned to engage in Problem 3. Give undergraduates time to first discuss Problem 3(a) in small groups.

Class Activity Problem 3 : Part a

3. Based on his work in Problem 1, Omar guesses that, in $\mathbb{Z}_{10}$, the equation $0 = 3x$ will have multiple solutions.

   (a) Why do you think Omar might have made this hypothesis?

   Sample Responses:

   - Omar might think that equations in $\mathbb{Z}_{10}$ behave more like equations in $\mathbb{Z}_6$ because both 6 and 10 are even numbers, unlike 5.
   - Omar sees that both 10 and 6 are composite numbers, unlike 5, and assumes that equations

will behave similarly in both $\mathbb{Z}_{10}$ and $\mathbb{Z}_6$.
- Omar might think that any modulus larger than 5 will have multiple solutions.

Discuss the following connection to teaching:

Discuss This Connection to Teaching

Problem 3(a) requires undergraduates to consider a hypothesis offered by a hypothetical student and to attempt to determine the mathematical reasoning that the student may have used to reach their hypothesis. Building capacity for considering the mathematical thinking of others strengthens undergraduates' own mathematical understanding and develops skills that they can apply in many settings, especially in the work of teaching, that require analyzing and valuing the thinking of others. Prospective teachers should be able to respond to student thinking when considering a reasonable, but incorrect, student answer by determining and addressing plausible reasoning trajectories, drawing out their conceptions and building on their understandings.

Ask undergraduates to work on Problems 3(b) and 3(c) in their groups. Think about the conclusions undergraduates offer in their groups and decide on the order in which you will have groups report out in a whole-class discussion. During whole-class discussion, ensure consensus is reached on the answers to Problems 3(b) and 3(c) before moving on to Problem 3(d).

Class Activity Problem 3 : Parts b & c

(b) In $\mathbb{Z}_{10}$, for which non-zero value(s) of a does the equation $0 = ax$ have a unique solution? Was Omar's hypothesis correct?

Solution:
For $a \in \{1, 3, 7, 9\}$ the solution is unique. This means Omar's hypothesis was not correct.

(c) In $\mathbb{Z}_{10}$, for which non-zero value(s) of a does the equation $1 = ax$ have a solution?

Solution:
For $a \in \{1, 3, 7, 9\}$ a solution exists.

Commentary:
Point out that Pre-Activity Problem 1(b) will help significantly with Problem 3(c) here. As you circulate the classroom, consider using the following prompts to generate discussion:

- How might you use a graph to help answer these questions?
- For Problem 3(c), will there ever be more than one solution for a given value of a? Why or why not?

Problems 3(b) and 3(c) are essentially asking undergraduates to find all the zero divisors and units in $\mathbb{Z}_{10}$, respectively. However, by framing these problems as solving equations, we reinforce the idea that an equation can be thought of as a property that applies precisely to elements of the solution set.

Problem 3(d) can be posed to the class at large and discussed without requiring undergraduates to work in groups first.

Class Activity Problem 3 : Part d

(d) Look back at your answers to Problems 3(b) and 3(c). What relationship do these integers have with 10, the modulus of $\mathbb{Z}_{10}$?

Solution:
The elements of $\mathbb{Z}_{10}$ that correctly answer 3(b) and 3(c) are those elements $x \in \mathbb{Z}_{10}$ for which $\gcd(x, 10) = 1$.

Discussion: The Structure of the Ring of Integers Modulo *n* (15 minutes)

To tie together all the parts of Problem 3, define the following vocabulary using the notation in your classroom textbook:

- Zero divisors.
- Integral domains.

Ask undergraduates for a conjecture about which elements of $\mathbb{Z}_{10}$ are zero divisors and which are units based on their work in Problem 3. To formalize this conjecture, introduce the following three theorems. As appropriate for your class (and considering time constraints), you may choose to prove some or all of the theorems in class.

Theorem 1. *Every nonzero element of $\mathbb{Z}_n$ is either a unit or a zero divisor.*

Proof. Let a be some nonzero element of $\mathbb{Z}_n$ and consider the set S of elements $\{0a, 1a, 2a \dots, (n-1)a\}$. If S has n distinct elements, then each one must match a unique element from $\mathbb{Z}_n = \{0, 1, 2, \dots, n-1\}$. So, $ba = 1$ for some b in $\mathbb{Z}_n$; since $\mathbb{Z}_n$ is also a commutative ring, $ab = 1$ as well and so a is a unit. If, on the other hand, S fails to contain n distinct elements, we must have $ba = ca$ for two distinct elements b and c in $\mathbb{Z}_n$. Consequently, $(b - c)a = 0$ and a is a zero divisor. □

Theorem 2. *An element a of $\mathbb{Z}_n$ is a unit if and only if it is relatively prime with n.*

Proof. Let $a \in \mathbb{Z}_n$ be a unit. Then, for some $b \in \mathbb{Z}_n$, $ab = 1$. If we interpret this equation as a statement about integers (not equivalence classes of integers), this means that ab and 1 differ by some multiple of n; that is, we have that $ab + kn = 1$ for some integer k. Consequently, if a and n have any common factor, then it must divide 1. Thus, $\gcd(a, n) = 1$.

Conversely, let $a \in \mathbb{Z}_n$ be nonzero and $\gcd(a, n) = 1$. Assume, for the sake of contradiction, that a is not a unit of $\mathbb{Z}_n$. Then, by the previous result, it is a zero divisor of $\mathbb{Z}_n$ and so there is a nonzero element b such that $ab = 0$. If we interpret this equation as a statement about integers (not equivalence classes of integers), this means that ab is some multiple of n; that is, we have that $ab = kn$. This implies that $n|ab$, but because $\gcd(a, n) = 1$, it must be that $n|b$. This forces b to be zero, which is a contradiction. So our assumption cannot hold—and thus a is a unit of $\mathbb{Z}_n$. □

Theorem 3. *$\mathbb{Z}_n$ is both a field and an integral domain when n is prime, but neither when n is composite.*

Proof. If n is prime, for every nonzero element $a \in \mathbb{Z}_n$, $\gcd(a, n) = 1$. By Theorem 2, this implies that all the nonzero elements of the ring are units. Then, by Theorem 1, this implies that no nonzero elements of the ring are zero divisors. Hence, $\mathbb{Z}_n$ is both a field and an integral domain. On the other hand, if n is composite, it is clear that any of its factors are zero divisors. These elements cannot be units, so $\mathbb{Z}_n$ is neither a field nor an integral domain. □

To proceed, ensure that undergraduates understand the flow of ideas across these three theorems. The following explanation may help facilitate this understanding: If a and n are relatively prime, then a is a unit in $\mathbb{Z}_n$. This means it has a unique multiplicative inverse a^{-1}. Because the inverse is unique, $ax = b$ must have a unique solution in $\mathbb{Z}_n$: namely, $x = a^{-1}b$. When $b = 0$, this shows that a is also not a zero divisor.

You might also wish to show that no element of any ring can ever be both a unit and a zero divisor: Assume $a \neq 0$ is both a unit and a zero divisor. Then, $\exists\, b \neq 0$ such that $ab = 0$ and $b = 1b = (a^{-1}a)b = a^{-1}(ab) = 0$, which is a contradiction.

Class Activity: Problems 4–6 (20 minutes)

Before asking your class to begin Problem 4, initiate a brief discussion by asking undergraduates what familiar techniques we use in $\mathbb{R}$ to solve for the roots of quadratic expressions and record their answers (on the board, at the document camera, or post their answers). Then, ask undergraduates to work in small groups on Problem 4. While they work, ask them to think about which of the recorded techniques may no longer work as expected. Also, have them take note of new methods that can be used that would not apply when working in $\mathbb{R}$.

Consider asking any of the following prompts to promote discussion as you circulate the classroom:

- How do you expect finding the roots of this equation in $\mathbb{Z}_{10}$ to be similar to or different from working in $\mathbb{R}$?
- Which of these methods seems the most difficult to use in $\mathbb{Z}_{10}$? Easiest? Why?
- Make a prediction about the number of solutions to this equation.

Ask the groups to share out and adjust the list accordingly to reflect what methods may or may not work in $\mathbb{Z}_{10}$. If no one brought it up, point out that for small enough finite rings it is completely valid to check every element individually. Add this method to the previously compiled list.

Class Activity Problem 4

For Problems 4–6, consider the quadratic equation $x^2 - 5x + 6 = 0$ in $\mathbb{Z}_{10}$.

4. What are some ways that you might attempt to solve this equation for x?

   Sample Responses:

   - Factoring or applying the quadratic formula.
   - Completing the square.
   - Graphing the equation and looking for its roots.
   - Checking all the elements of $\mathbb{Z}_{10}$ to see which of them solve the equation.

Encourage undergraduates to find the proposed solution and others in Problem 5(a) using whatever method they like from the new list. Allow them to work in small groups for both parts of Problem 5.

Class Activity Problem 5

5. Notice that we can factor the left-hand side of this equation to obtain $(x-2)(x-3) = 0$, from which we find that $x - 2 = 0$ or $x - 3 = 0$. This yields the solutions $x = 2$ and $x = 3$.

   (a) Verify that $x = 7$ is also a solution. Are there any more? Why do you think factoring did not yield *all* the solutions?

   Solution:
   Because $7^2 - 5 \cdot 7 + 6 = 49 - 35 + 6 = 9 - 5 + 6 = 10 = 0$, 7 is a solution. We can also see that 8 is a solution: $8^2 - 5 \cdot 8 + 6 = 64 - 40 + 6 = 4 - 0 + 6 = 10 = 0$. Factoring did not yield all solutions because, as we saw in Problem 1, sometimes rings contain zero divisors. If so, we can't claim that $a \cdot b = 0 \Rightarrow a = 0$ or $b = 0$.

   (b) What important property of $\mathbb{R}$ are we using when we find the roots of a factored expression and claim those roots constitute the entire solution set?

   Solution:
   The zero product property.

For Problem 5(a), allow for some informality in undergraduates' responses since Problem 5(a) aims to elicit student thinking that scaffolds their more formal responses to Problem 5(b). When discussing Problem 5(b), discuss the following connection to teaching:

Discuss This Connection to Teaching

Solving quadratic equations in one variable is an ubiquitous standard in high school algebra. By attempting to reproduce familiar root-finding techniques outside of $\mathbb{R}$, prospective teachers are able to identify aspects of solving polynomial expressions that high school students might take for granted. That is, in high school, $\mathbb{R}$ has the *zero product property* only because it is an integral domain; similarly, every element of $\mathbb{R}$ has a multiplicative inverse only because $\mathbb{R}$ is also a field. Considering equation-solving in these other settings enables prospective teachers an anchor for applying reasoning that follows from $\mathbb{R}$ being a field and provides a foundation for looking forward to the introduction of other algebraic structures that may not have these familiar properties. This look back to equation-solving in high school underscores the important ideas that support the processes taught for solving equations.

There are ways to possibly determine additional solutions to a quadratic equation via the factoring technique, such as rewriting the coefficients of a given quadratic expression via equivalent elements of the ring. In this problem, $x^2 - 5x + 6 = x^2 + 5x + 6$ in $\mathbb{Z}_{10}$. So, we can also say $x^2 + 5x + 6 = 0 \Rightarrow (x + 2)(x + 3) = 0$, and so $x = -2 = 8$ or $x = -3 = 7$.

Another way to use factoring to find the entire solution set is, instead of setting each factor equal to zero, setting each factor equal to a pair of zero divisors. Then, if both factors give the same x value, that is a solution. For example, $x - 2 = 5$ and $x - 3 = 4$ both give $x = 7$, so this is a solution. On the other hand, $x - 2 = 5$ and $x - 3 = 8$ give $x = 7$ and $x = 1$ respectively, so we can make no claims about whether 7 or 1 are additional solutions.

Finally, give undergraduates a few minutes to discuss Problem 6 in small groups.

Class Activity Problem 6

6. Attempt to apply the quadratic formula to the above equation. What difficulties do you encounter?

   Sample Responses:

   - Modular arithmetic is more difficult than the usual arithmetic.
   - The quadratic formula requires us to "divide by 2," which does not make sense in rings where 2 is not a unit.
   - Taking square roots is difficult in general—even in a field, there is no guarantee that a particular element will even have a square root. In non-fields, certain elements may have a different number of square roots than we might expect.

Call the class back together for a whole-class discussion. Give groups a chance to report out first on their successes and obstacles encountered while attempting Problem 6. The most prominent point to raise is that the quadratic formula relies on multiplication by $1/2a$, which may not be well-defined if $2a$ is not a unit.

Like factoring, the quadratic formula is not a lost cause. In this case, we can write $2x = 5 + \sqrt{25 - 24} \Rightarrow 2x = 5 + \sqrt{5 + 6} \Rightarrow 2x = 5 + \sqrt{1}$, which yields two equations: $2x = 5 + 1 = 6$ and $2x = 5 + 9 = 4$. It can be seen that both equations have two solutions, resulting in the complete solution set of $\{2, 3, 7, 8\}$.

Wrap-Up (5 Minutes)

Recap the lesson briefly for the class:

- Some rings have zero divisors; those that don't are called integral domains and have the zero product property, which we rely upon when applying familiar processes for solving for the roots of a quadratic equation.

- $\mathbb{Z}_p$ is an integral domain and a field for p prime.

When working outside of a field or an integral domain to solve linear or quadratic equations, certain techniques that we learned in high school (such as factoring or the quadratic formula) become unreliable or much more difficult to apply.

We end this lesson with the use of an exit ticket. See Chapter 1 for advice about concluding mathematics lessons using exit tickets.

Homework Problems

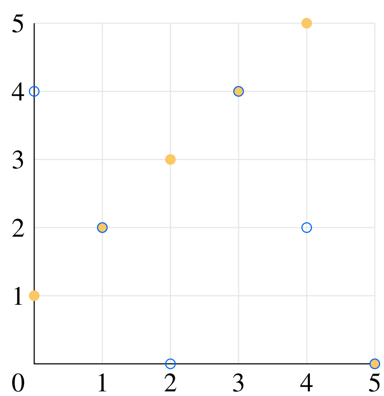
At the end of the lesson, assign the following homework problems.

Besides finding solutions to linear equations, another common task in high school algebra is the converse: given two points which lie on a line, find the corresponding equation to which they are both solutions. Problem 1 illustrates again that working outside of a field can have unexpected consequences.

Homework Problem 1

1. In $\mathbb{R} \times \mathbb{R}$, for any two distinct points A and B, there exists a unique line containing them. Show this statement is not true in $\mathbb{Z}_6 \times \mathbb{Z}_6$ by finding the equations of two distinct lines that both contain the points $(1, 2)$ and $(3, 4)$. [Recall that for a set A of elements (numbers) with a well-defined notion of addition and multiplication, we define a *line over A* as the solution set to an equation of the form $ax + by = c$ for some fixed values of $a, b, c \in \text{A}$. That is, a line is the set of all ordered pairs $(x, y) \in \text{A} \times \text{A}$ that make $ax + by = c$ a true statement in A. The *graph of a line* is a scatter plot of the solution set on a coordinate plane, usually one with perpendicular axes marked by the elements of A.]

   Solution:
   The lines $y = x + 1$ and $y = 4x + 4$ are two examples, illustrated below.



   More generally, a line $ax + by = c$ in $\mathbb{Z}_6 \times \mathbb{Z}_6$ containing the points $(1, 2)$ and $(3, 4)$ must satisfy $a + 2b = c$ and $3a + 4b = c$, so $a + 2b = 3a + 4b \Rightarrow 4a + 4b = 0 \Rightarrow 4(a + b) = 0 \Rightarrow a + b \in \{0, 3\}$. If $a + b = 0$, then $a + 2b = c \Rightarrow (a + b) + b = c \Rightarrow b = c$, so $a + c = 0$ and $c = 5a$. Similarly, if $a + b = 3$, then $a + 2b = c \Rightarrow (a + b) + b = c \Rightarrow 3 + b = c \Rightarrow b = c + 3$, so $a + 3 + c = 3 \Rightarrow c = 5a$. That is, any line $ax + by = 5a$ where $a + b \in \{0, 3\}$ will work.

Problem 2 prompts undergraduates to examine and guide another student's thinking which helps them grow their own understanding of the topic. Here, the undergraduates learn that the structure of a ring does not necessarily transfer to a direct product of that ring with itself.

Homework Problem 2

2. Artyom says that since $\mathbb{R}$ is an integral domain, then the set of ordered pairs $\mathbb{R} \times \mathbb{R}$ must also be an integral domain under the operations given below:

$$(a, b) \oplus (c, d) = (a + c, b + d)$$
$$(a, b) \otimes (c, d) = (a \cdot c, b \cdot d)$$

   (a) Why is Artyom incorrect?

   Solution:
   While $\mathbb{R} \times \mathbb{R}$ is a ring under the operations described above, it is not an integral domain. For $a, d \neq 0$, it is clear that neither $(a, 0)$ nor $(0, d)$ are "equal to zero". That is, neither is the additive identity. Then, $(a, 0) \otimes (0, d) = (0, 0)$ implies that $(a, 0)$ and $(0, d)$ are zero divisors.

   (b) What question would you ask Artyom to help him understand his error? Why would your question be helpful?

   Sample Responses:
   - Can you find a pair of elements in $\mathbb{R} \times \mathbb{R}$ whose product has at least one component that is zero? Hopefully, this question will lead Artyom to consider if he could simultaneously make the other component equal to zero, leading to zero divisors.
   - Can a nonzero element of $\mathbb{R} \times \mathbb{R}$ still contain a zero as one of its components? With this question, I want to lead Artyom towards possible counterexamples.

A useful alternative definition for integral domains (a ring is an integral domain if and only if cancellation law holds) is introduced and proven in Problem 3.

Homework Problem 3

3. Let R be a commutative ring in which the multiplicative identity and additive identity are distinct elements.

   (a) Prove that if R is an integral domain, then for $a, b, c \in$ R and $a \neq 0$, $a \cdot b = a \cdot c \Rightarrow b = c$.

   Solution:
   Let $a \neq 0$. Then, $a \cdot b = a \cdot c \Rightarrow a \cdot b - a \cdot c = 0 \Rightarrow a \cdot (b - c) = 0$. R is an integral domain and $a \neq 0$, so by the fact that R contains no zero divisors we conclude that $b - c = 0 \Rightarrow b = c$.

   (b) Prove that if $\forall\, a, b, c \in$ R with $a \neq 0$ we have that $a \cdot b = a \cdot c \Rightarrow b = c$, then R is an integral domain.

   Solution:
   Let $a \neq 0$. Let $a \cdot b = 0$. If we show that $b = 0$, then we have established that R has no zero divisors and is an integral domain. But $a \cdot b = 0 \Rightarrow a \cdot b = a \cdot 0$. By hypothesis, we now conclude $b = 0$.

After dealing with linear and quadratic functions in the Class Activity, a natural next step is to look for ways to solve simple cubic functions, as presented in Problem 4.

Homework Problem 4

4. When looking for solutions to the equation $x^3 = 1$ for $x \in \mathbb{Z}_{13}$, we see that $x = 1$ clearly works. To find other solutions, it might be useful to observe that every element in $\mathbb{Z}_{13}$ corresponds to a value 2^n

for some n by completing the following table of values in $\mathbb{Z}_{13}$. [Hint: Double the values in the table from left to right, remembering to convert to modulo 13 when appropriate]

2^0	2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}	2^{11}	2^{12}
1	2	4	8	3	6	12	11	9	5	10	7	1

Now, to find other solutions, we might use the table above to help; for example, $x^3 = 1 = 2^{12} = (2^4)^3 = 3^3$. Thus, 3 is also a solution. It turns out there is only one more solution to this equation. Find it and justify your answer by using powers of 2.

Solution:
The remaining answer is 9: $x^3 = 1 = 1^2 = (2^{12})^2 = 2^{24} = (2^8)^3 = 9^3$.

In fact, we can show that there are no other solutions. Given that every element of $\mathbb{Z}_{13}$ corresponds to a value of 2^n for some n. Then, if $x = 2^n$ is a solution to $x^3 = 1$, we have that $2^{3n} = 1$. From the table, this means that $3n$ must be a multiple of 12 and so n is a multiple of 4. Then: $x = 2^n = 2^{4k} = 16^k = 3^k$, where k is some non-negative integer. Plugging in values of k, we see that the only solutions are the three we have found.

In the Class Activity, undergraduates learned when the equation $ax = 0$ has a unique solution in $\mathbb{Z}_n$. Problem 5 broadens that understanding, such that undergraduates are able to describe the number of solutions to $ax = 0$ in $\mathbb{Z}_n$ when it is not unique.

Homework Problem 5

5. How many solutions does the equation $ax = 0$ have in $\mathbb{Z}_{12}$ for each nonzero a? Use your answer to make a hypothesis about the number of solutions to the equation $ax = 0$ in $\mathbb{Z}_n$ when a is nonzero.

Sample Response:

a	0	1	2	3	4	5	6	7	8	9	10	11
#	X	1	2	3	4	1	6	1	4	3	2	1

I notice that the number of solutions to $ax = 0$ in $\mathbb{Z}_{12}$ is $\gcd(a, 12)$. I assume that this also holds in $\mathbb{Z}_n$.

In addition to solving linear and quadratic equations, high school students are often tasked with solving systems of equations. Problem 6 extends the connections to teaching made during the Class Activity to include systems of equations; that is, prospective teachers investigate how certain familiar techniques (in this case, the substitution or elimination methods) may or may not translate outside the field of real numbers.

Homework Problem 6

6. Solve the system of linear equations given below in the following rings, if possible.

$$2x + y = 4$$
$$x + 2y = 0$$

(a) $\mathbb{Z}_5 \times \mathbb{Z}_5$

Solution:
First, $x + 2y = 0 \Rightarrow x = 3y$. Substituting into the other equation, we have $2x + y = 4 \Rightarrow 2(3y) + y = 4 \Rightarrow 7y = 4 \Rightarrow 2y = 4$. Multiplying both sides by 3 yields $y = 12 = 2$. Then, $x = 3y \Rightarrow x = 6 = 1$, so $(1, 2)$ is a unique solution.

(b) $\mathbb{Z}_6 \times \mathbb{Z}_6$

Solution:
First, $x + 2y = 0 \Rightarrow x = 4y$. Substituting into the other equation, we have $2x + y = 4 \Rightarrow 2(4y) + y = 4 \Rightarrow 9y = 4 \Rightarrow 3y = 4$. But $\gcd(3, 6)$ is not 1, so this equation does not have a unique solution. In fact, since 3 has no multiplicative inverse in $\mathbb{Z}_6$, $3y = 4$ has no solution and neither does the system of equations.

(c) Was your process for solving parts (a) and (b) the same? Why or why not? What difficulties arose in parts (a) and (b)?

Sample Responses:

- I tried to answer both part (a) and part (b) algebraically, but was only successful in part (a). In part (b), the fact that 3 has no multiplicative inverse in $\mathbb{Z}_6$ prevented me from solving the equation.
- Graphing both lines on a coordinate axis reveals a point of intersection in $\mathbb{Z}_5 \times \mathbb{Z}_5$, but the scatter plots don't overlap in $\mathbb{Z}_6 \times \mathbb{Z}_6$.

Assessment Problems

The following two problems address ideas explored in the lesson, with a focus on connections to teaching and mathematical content. You can include these problems as part of your usual course quizzes or exams.

Assessment Problems 1 & 2

1. List all the nonzero values of a which give the equation $ax = 0$ a unique solution in the following rings:

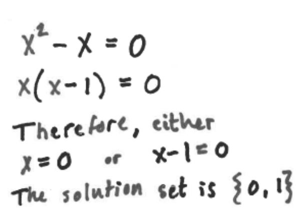
   (a) $\mathbb{Z}_{13}$

   Solution:
   Since 13 is prime, $\gcd(a, 13) = 1 \; \forall \, a \in \mathbb{Z}_{13}$. This means that $ax = 0$ has a unique solution $\forall \, a \in \mathbb{Z}_{13}$, so $a \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.

   (b) $\mathbb{Z}_{14}$

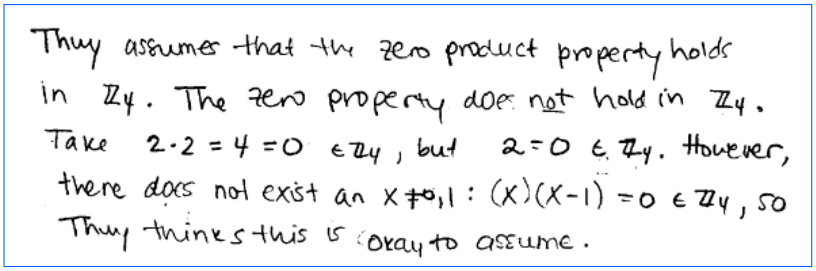
   Solution:
   We need only identify the elements of $\mathbb{Z}_{14}$ which are relatively prime with 14. These are the values of a for which $ax = 0$ will have a unique solution. So, $a \in \{1, 3, 5, 9, 11, 13\}$.

2. Thuy's work for finding solutions to $x^2 - x = 0$ in $\mathbb{Z}_4$ is shown below.



(a) From her work, what assumption does Thuy seem to be making about $\mathbb{Z}_4$? Is this assumption correct?

Sample Response:


Thuy assumes that the zero product property holds in $\mathbb{Z}_4$. The zero property does not hold in $\mathbb{Z}_4$. Take $2 \cdot 2 = 4 = 0 \in \mathbb{Z}_4$, but $2 = 0 \in \mathbb{Z}_4$. However, there does not exist an $x \neq 0,1 : (x)(x-1) = 0 \in \mathbb{Z}_4$, so Thuy thinks this is okay to assume.

(b) Thuy checks each element of $\mathbb{Z}_4$ and verifies that her solution set is correct. Her teacher asks her to attempt to solve the same equation, this time in $\mathbb{Z}_6$. What is the teacher hoping Thuy will understand about her approach by working in $\mathbb{Z}_6$?

Sample Response:

Thuy happened to find all the solutions in $\mathbb{Z}_4$ despite the fact that $\mathbb{Z}_4$ is not an integral domain, but she might not be so lucky with a different quadratic equation. If Thuy works the same problem the same way in $\mathbb{Z}_6$, she will not find all the solutions since 3 and 4 also solve the equation. This will help her see that she cannot (reliably) apply the zero product property in $\mathbb{Z}_n$ when n is composite.

1.6 References

[1] National Governors Association Center for Best Practices & Council of Chief State School Officers (2010). *Common Core State Standards for Mathematics.* Authors. Retrieved from `http://www.corestandards.org/`

1.7 Lesson Handouts

Handouts for use during instruction are included on the pages that follow. LaTeX files for these handouts can be downloaded from INSERT URL HERE.

NAME: **PRE-ACTIVITY:** SOLVING EQUATIONS (page 1 of 2)

1. Recall that $\mathbb{Z}_n$ is the set of *equivalence classes* on the integers, where two integers are in the same equivalence class if and only if they both have the same (smallest, non-negative) remainder when divided by n. The set $\mathbb{Z}_n$ contains n such equivalence classes which, canonically, are represented by the possible remainders when an integer is divided by n: $\{0, 1, \ldots, n-2, n-1\}$.

   (a) Fill in the following chart with the representative of each integer's equivalence class in $\mathbb{Z}_{10}$.

Integer	36	17	-4	-17
Representative in $\mathbb{Z}_{10}$				

   If we are careful, we can also (sometimes) represent non-integers as elements of $\mathbb{Z}_n$. For example, if we interpret the notation "$1/3$" as "the element you multiply by 3 to get 1," we would then consider 7 in $\mathbb{Z}_{10}$ a representative of "$1/3$", since $3 \cdot 7 = 21 = 1$ (where $21 = 1$ because 21 has remainder 1 when divided by 10). Furthermore, no other element of $\mathbb{Z}_{10}$ has this property.

   (b) Fill in the following chart with the representative in $\mathbb{Z}_{10}$, if it exists.

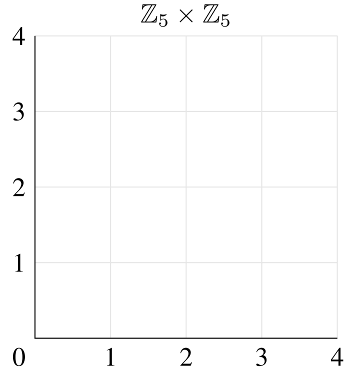
"Non-integer"	"$1/1$"	"$1/2$"	"$1/3$"	"$1/4$"	"$1/5$"	"$1/6$"	"$1/7$"	"$1/8$"	"$1/9$"
Representative in $\mathbb{Z}_{10}$	1		7						

2. Let A be a set of elements (numbers) with a well-defined notion of addition and multiplication. We define a *line over A* as the solution set to an equation of the form $ax + by = c$ for some fixed values of $a, b, c \in$ A. That is, a line is the set of all ordered pairs $(x, y) \in$ A $\times$ A that make $ax + by = c$ a true statement in A. The *graph of a line* is a scatter plot of the solution set on a coordinate plane, usually one with perpendicular axes marked by the elements of A.

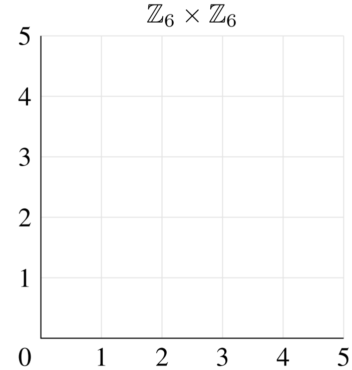
   For example, in the system of real numbers, the set of all ordered pairs that make $y = 3x$ a true statement in $\mathbb{R}$ has a graph which is a continuous straight geometric line of slope 3 through the point $(0, 0)$ in our usual Cartesian coordinate system.

   Graph the line $y = 3x$ in the following spaces on the provided axes.

$\mathbb{Z}_5 \times \mathbb{Z}_5$



$\mathbb{Z}_6 \times \mathbb{Z}_6$



3. In solving the equation $x + 4 = 1 + 4x$ in $\mathbb{R}$, a student makes the following algebraic manipulations:

$$\begin{aligned} x + 4 &= 1 + 4x \\ 4 &= 1 + 3x \\ 3 &= 3x \\ 1 &= x \end{aligned}$$

The student then concludes that $x = 1$ is the solution set to $x + 4 = 1 + 4x$ in $\mathbb{R}$.

(a) Describe the mathematical justification for each step in the student's solution.

(b) To solve $x + 4 = 1 + 4x$ for x in $\mathbb{Z}_5$, are we allowed to repeat the process the student used (in $\mathbb{R}$) as presented above? Does this process yield the entire solution set to the equation? Explain.

(c) To solve $x + 4 = 1 + 4x$ for x in $\mathbb{Z}_6$, are we allowed to repeat the process the student used (in $\mathbb{R}$) as presented above? Does this process yield the entire solution set to the equation? Explain.

NAME: **CLASS ACTIVITY:** SOLVING EQUATIONS (page 1 of 2)

Consider the linear equation $y = 3x$ (and your corresponding graphs) from the Pre-Activity.

1. How many solutions to $0 = 3x$ exist in the following domains? What are they?

Domain	$\mathbb{R}$	$\mathbb{Z}_5$	$\mathbb{Z}_6$
# of Sol.			
Sol. Set			

2. How many solutions to $1 = 3x$ exist in the following domains? What are they?

Domain	$\mathbb{R}$	$\mathbb{Z}_5$	$\mathbb{Z}_6$
# of Sol.			
Sol. Set			

3. Based on his work in Problem 1, Omar guesses that, in $\mathbb{Z}_{10}$, the equation $0 = 3x$ will have multiple solutions.

   (a) Why do you think Omar might have made this hypothesis?

   (b) In $\mathbb{Z}_{10}$, for which non-zero value(s) of a does the equation $0 = ax$ have a unique solution? Was Omar's hypothesis correct?

   (c) In $\mathbb{Z}_{10}$, for which non-zero value(s) of a does the equation $1 = ax$ have a solution?

   (d) Look back at your answers to Problems 3(b) and 3(c). What relationship do these integers have with 10, the modulus of $\mathbb{Z}_{10}$?

CLASS ACTIVITY: SOLVING EQUATIONS (page 2 of 2)

For problems 4–6, consider the quadratic equation $x^2 - 5x + 6 = 0$ in $\mathbb{Z}_{10}$.

4. What are some ways that you might attempt to solve this equation for x?

5. Notice that we can factor the left-hand side of this equation to obtain $(x-2)(x-3) = 0$, from which we find that $x - 2 = 0$ or $x - 3 = 0$. This yields the solutions $x = 2$ and $x = 3$.

   (a) Verify that $x = 7$ is also a solution. Are there any more? Why do you think factoring did not yield *all* the solutions?

   (b) What important property of $\mathbb{R}$ are we using when we find the roots of a factored expression and claim those roots constitute the entire solution set?

6. Attempt to apply the quadratic formula to the above equation. What difficulties do you encounter?

NAME: **HOMEWORK PROBLEMS:** SOLVING EQUATIONS (page 1 of 1)

1. In $\mathbb{R} \times \mathbb{R}$, for any two distinct points A and B, there exists a unique line containing them. Show this statement is not true in $\mathbb{Z}_6 \times \mathbb{Z}_6$ by finding the equations of two distinct lines that both contain the points $(1, 2)$ and $(3, 4)$. [Recall that for a set A of elements (numbers) with a well-defined notion of addition and multiplication, we define a *line over A* as the solution set to an equation of the form $ax + by = c$ for some fixed values of $a, b, c \in$ A. That is, a line is the set of all ordered pairs $(x, y) \in$ A $\times$ A that make $ax + by = c$ a true statement in A. The *graph of a line* is a scatter plot of the solution set on a coordinate plane, usually one with perpendicular axes marked by the elements of A.]

2. Artyom says that since $\mathbb{R}$ is an integral domain, then the set of ordered pairs $\mathbb{R} \times \mathbb{R}$ must also be an integral domain under the operations given below:

$$(a, b) \oplus (c, d) = (a + c, b + d)$$
$$(a, b) \otimes (c, d) = (a \cdot c, b \cdot d)$$

   (a) Why is Artyom incorrect?
   (b) What question would you ask Artyom to help him understand his error? Why would your question be helpful?

3. Let R be a commutative ring in which the multiplicative identity and additive identity are distinct elements.
   (a) Prove that if R is an integral domain, then for $a, b, c \in$ R and $a \neq 0$, $a \cdot b = a \cdot c \Rightarrow b = c$.
   (b) Prove that if $\forall\, a, b, c \in$ R with $a \neq 0$ we have that $a \cdot b = a \cdot c \Rightarrow b = c$, then R is an integral domain.

4. When looking for solutions to the equation $x^3 = 1$ for $x \in \mathbb{Z}_{13}$, we see that $x = 1$ clearly works. To find other solutions, it might be useful to observe that every element in $\mathbb{Z}_{13}$ corresponds to a value 2^n for some n by completing the following table of values in $\mathbb{Z}_{13}$. [Hint: Double the values in the table from left to right, remembering to convert to modulo 13 when appropriate]

2^0	2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}	2^{11}	2^{12}
				3								1

   Now, to find other solutions, we might use the table above to help; for example, $x^3 = 1 = 2^{12} = (2^4)^3 = 3^3$. Thus, 3 is also a solution. It turns out there is only one more solution to this equation. Find it and justify your answer by using powers of 2.

5. How many solutions does the equation $ax = 0$ have in $\mathbb{Z}_{12}$ for each nonzero a? Use your answer to make a hypothesis about the number of solutions to the equation $ax = 0$ in $\mathbb{Z}_n$ when a is nonzero.

6. Solve the system of linear equations given below in the following rings, if possible.

$$2x + y = 4$$
$$x + 2y = 0$$

   (a) $\mathbb{Z}_5 \times \mathbb{Z}_5$
   (b) $\mathbb{Z}_6 \times \mathbb{Z}_6$
   (c) Was your process for solving parts (a) and (b) the same? Why or why not? What difficulties arose in parts (a) and (b)?

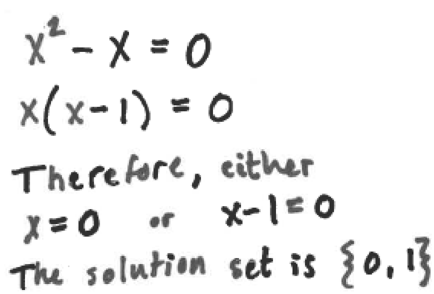
NAME: **ASSESSMENT PROBLEMS:** SOLVING EQUATIONS (page 1 of 1)

1. List all the nonzero values of a which give the equation $ax = 0$ a unique solution in the following rings:

   (a) $\mathbb{Z}_{13}$

   (b) $\mathbb{Z}_{14}$

2. Thuy's work for finding solutions to $x^2 - x = 0$ in $\mathbb{Z}_4$ is shown below.



   (a) From her work, what assumption does Thuy seem to be making about $\mathbb{Z}_4$? Is this assumption correct?

   (b) Thuy checks each element of $\mathbb{Z}_4$ and verifies that her solution set is correct. Her teacher asks her to attempt to solve the same equation, this time in $\mathbb{Z}_6$. What is the teacher hoping Thuy will understand about her approach by working in $\mathbb{Z}_6$?