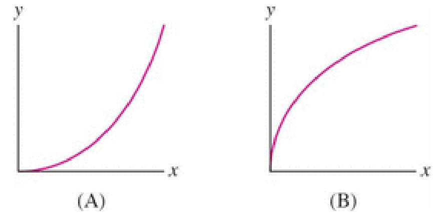
You can check your answers in WebWork. Full solutions in WW available Tuesday evening.

Problem 1. Which graph has the property: For all x, the average rate of change over the interval $[0, x]$ is greater then the instantaneous rate of change at x.



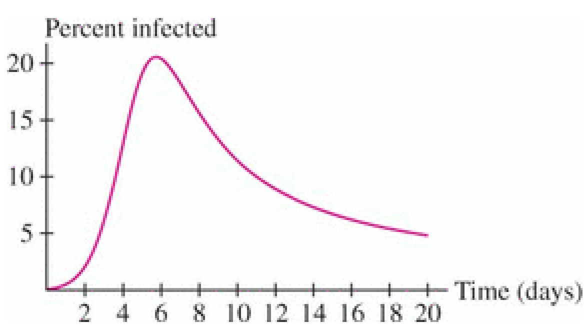
Problem 2. Find the average rate of change of the given function over the given interval. Express your answers in terms of square roots and π, do not give decimal expressions.

a. $\sin(x)$ over $0 \leq x \leq \pi/4$

b. $\cos(x)$ over $\pi/6 \leq x \leq \pi/3$

c. Is there an interval over which the functions $\sin(x)$ and $\cos(x)$ have the same average rate of change that is non-zero? (Hint: Consider the graphs of these functions over one whole cycle, e.g. for $0 \leq x \leq 2\pi$. Where do they intersect?)

Problem 3. An epidemiologist finds that the percentage N (t) of susceptible children who were infected on day t during a measles outbreak is given, to a reasonable approximation, by the formula: $N(t) = \frac{100t^2}{t^3+5t^2-100t+380}$



a. Is the rate of decline greater at $t = 8$ or $t = 16$?

b. Draw the secant line whose slope is the average rate of change in infected children over the interval $[0, 4]$.

c. Estimate $\Delta N/\Delta t$ on the interval $[0, 6]$. What are the units of $\Delta N/\Delta t$.

d. Using the formula for $N(t)$, compute $\lim_{t\to\infty} N(t)$.

Problem 4. In the theory of relativity, the mass of a particle with velocity v is:

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where m_0 is the mass of the particle at rest and c is the speed of light. What happens as $v \to c^-$?

Problem 5. A right circular cylinder with a height of 10 cm and a surface ares of S cm^2 has a radius given by

$$r(S) = \frac{1}{2}\left(\sqrt{100 + \frac{2S}{\pi}} - 10\right).$$

Find $\lim_{S\to 0^+} \frac{r(S)}{S}$.

Problem 6. Determine the one-sided limits at $c = 1, 2, 3$ for the function f shown in the figure and state whether the limit exists at these points.

(a) $\lim\limits_{x \to 1^-} f(x) =$

(b) $\lim\limits_{x \to 1^+} f(x) =$

(c) $\lim\limits_{x \to 2^-} f(x) =$

(d) $\lim\limits_{x \to 2^+} f(x) =$

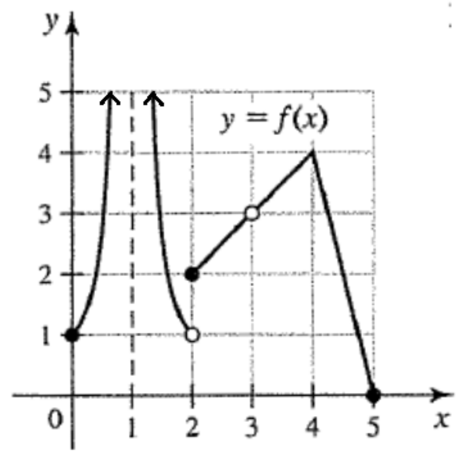
(e) $\lim\limits_{x \to 3^-} f(x) =$

(f) $\lim\limits_{x \to 3^+} f(x) =$

(g) Does the limit exist at $c = 1$? Yes/No

(h) Does the limit exist at $c = 2$? Yes/No

(i) Does the limit exist at $c = 3$? Yes/No



Problem 7. Given $\lim\limits_{x \to 2} f(x) = 5$ and $\lim\limits_{x \to 2} g(x) = 2$, use limit laws to compute the following limits or explain why we cannot find the limit. Make sure to keep the $\lim\limits_{x \to a}$ operator until the very last step.

1. $\lim\limits_{x \to 2}(2f(x) - g(x))$
2. $\lim\limits_{x \to 2}(f(x)g(2))$
3. $\lim\limits_{x \to 2} \dfrac{f(x)g(x)}{x}$
4. $\lim\limits_{x \to 2} f(x)^2 + x \cdot g(x)^2$
5. $\lim\limits_{x \to 2} [f(x)]^{\frac{3}{2}}$
6. $\lim\limits_{x \to 2} \dfrac{f(x) - 5}{g(x) - 2}$

Problem 8. Let c be a number and consider the function $f(x) = \begin{cases} cx^2 - 5 & \text{if } x < 1 \\ 10 & \text{if } x = 1 \\ \dfrac{1}{x} - 2c & \text{if } x > 1 \end{cases}$.

1. Find all numbers c such that $\lim\limits_{x \to 1} f(x)$ exists.
2. Is there a number c such that $f(x)$ is continuous at $x = 1$? Justify your answer.

Problem 9. For each limit, evaluate the limit or or explain why it does not exist.

1. $\lim\limits_{x \to 2} \dfrac{x - 2}{x^2 - 4}$
2. $\lim\limits_{x \to 2} \dfrac{x - 2}{\frac{1}{x} - \frac{1}{2}}$
3. $\lim\limits_{x \to 0^+} \dfrac{1}{\sqrt{x}} - \dfrac{1}{\sqrt{x^2 + x}}$
4. $\lim\limits_{t \to 0} \left(\dfrac{e^{2t} + 4e^t - 5}{e^t - 1} \right)$
5. $\lim\limits_{x \to 4} \dfrac{x^2 - 16}{\sqrt{x} - 2}$
6. $\lim\limits_{x \to 0} \dfrac{(2 + x)^2 - 4}{x}$

The following identity may be useful for the next problems.

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y) \tag{1}$$

Problem 10.

Use equation (1), to simplify and compute the limit

$$\lim_{h\to 0} \frac{\cos(x+h) - \cos(x)}{h}$$

Problem 11. Evaluate the following limits

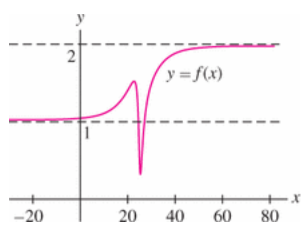
1. For $a \neq 0$, $\lim_{t\to 0} \dfrac{\sin(at)}{t}$
2. For $a \neq 0$ $\lim_{t\to 0} \dfrac{\sin^2(at)}{t}$
3. $\lim_{t\to \pi} \dfrac{\sin(t)}{t}$
4. $\lim_{x\to 0} \dfrac{\tan(2x)}{x}$
5. $\lim_{h\to 0} \dfrac{1-\cos(2h)}{\sin^2(3h)}$ **Hint:** Multiply by $\frac{1+\cos(2h)}{1+\cos(2h)}$

Problem 12. Let g be a function such that, for all real numbers x near 5 but not equal to 5:

$$4\cos(x-5) \leq g(x) \leq \frac{1}{5}x + \frac{5}{x} + 2.$$

Argue that $\lim_{x\to 5} g(x)$ exists and find its value. **As usual, justify your answer**.

Problem 13. Consider the graph of f below



What is $\lim_{x\to\infty} f(x)$?

What is $\lim_{x\to -\infty} f(x)$?

Does this function have any horizontal asymptotes? If so, what are they?

Problem 14. Find the horizontal asymptote(s) of $f(x) = \dfrac{2e^x + 3e^{2x}}{e^{2x} + e^{3x}}$.

Problem 15. Evaluate the following limits

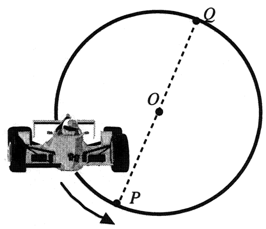
1. $\lim_{x\to\infty} \dfrac{x+2}{x^2-4}$
2. $\lim_{x\to -\infty} \dfrac{x-2}{x^2-4}$
3. $\lim_{x\to\infty} \dfrac{x-2}{\sqrt{x^2+4}}$
4. $\lim_{t\to -\infty} \dfrac{t-2}{\sqrt{t^2+4}}$
5. $\lim_{t\to -\infty} \dfrac{t^3+9t}{10t+3}$
6. $\lim_{t\to\infty} \dfrac{t^{4/3}+t^{1/3}}{(2t^{2/3}+2)^2}$

Problem 16. The hill function is given by

$$H(S) = \frac{S^n}{K^n + S^n}$$

where n and K are positive constants. Find $\lim_{S\to\infty} H(S)$.

Problem 17. A sprint car finishes a one-mile circular track in 20 seconds from a resting start. Show that during some 10 second interval, his car must pass through two points P and Q which lie opposite to each other on the track. Assume that the car moves with a varying but continuous speed. Hint: Let $\theta(t)$ be the angular position of the car at time t. For example $\theta(0) = 0$ and $\theta(20) = 2\pi$. Then consider Consider $G(t) = \theta(t + 10) - \theta(t) - \pi$.



a. $G(0) =$

b. $G(10) =$

c. What theorem from calculus can you use to conclude that G must have a zero and what must you assume?

Problem 18.

a. Explain in detail how you would use the intermediate value theorem to show that the equation $xe^x = 2$ has a solution in the interval $(0, 1)$. (Note: $e \approx 2.7$) Determine if the solution lies in the interval $(0, 1/2)$ or $(1/2, 1)$.

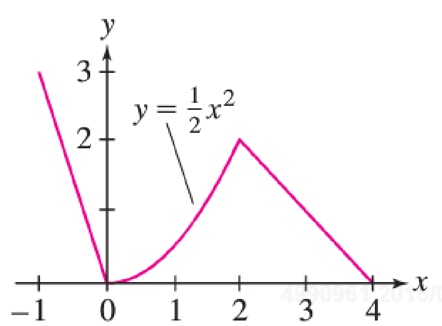
b. Suppose that h is a continuous function and has the following values

t	-1	1	-0.1	0.1	-0.01	0.01
$h(t)$	2	.5	3	2	-0.3	-2

List all of the smallest intervals for which h must have a root (zero).

Problem 19. Suppose that f is a continuous function on $[0, 1]$ such that $0 < f(x) < 1$ for every x in $[0, 1]$. Show that there is a number z in $[0, 1]$ such that $f(z) = z$. This is called a fixed point of f.

Problem 20. Consider the graph below of the function $f(x)$ on the interval $(-1, 4)$.



1. For which x values would the derivative $f'(x)$ not be defined?
2. Sketch the graph of the derivative function f'.

Problem 21. Given $f(x) = \dfrac{1}{3x+1}$, find $f'(1)$ using the limit definition of the derivative.

Problem 22. Given that $\lim_{a\to 0} \dfrac{\ln(1+a)}{a} = 1$, use the limit definition of the derivative to compute $f'(x)$ for $f(x) = \ln(x)$.