You can check your answers in WebWork. Full solutions in WW available Sunday evening.

**Problem 1.** Find the average rate of change of the given function over the given interval. Express your answers in terms of square roots and \( \pi \), do not give decimal expressions.

a. \( \sin(x) \) over \( 0 \leq x \leq \pi/4 \)

b. \( \cos(x) \) over \( \pi/6 \leq x \leq \pi/3 \)

c. Is there an interval over which the functions \( \sin(x) \) and \( \cos(x) \) have the same average rate of change that is non-zero? (Hint: Consider the graphs of these functions over one whole cycle, e.g. for \( 0 \leq x \leq 2\pi \). Where do they intersect?)

**Problem 2.** Use the tuna swimming data in figure below to answer the following questions

a. Determine the average velocity of each of these two fish over the 35 hours shown in the figure.

b. What is the fastest average velocity shown in this figure, and over what time interval and for which fish did it occur?

![Tuna Swimming Data](image)

**Problem 3.**

The graphs in the figure represent the positions \( s \) of moving particles as functions of time \( t \). Match each graph with a description:

a. Speeding up

b. Speeding up and then slowing down

c. Slowing down

d. Slowing down and then speeding up
Problem 4. Determine the one-sided limits at \( c = 1, 2, 3 \) for the function \( f \) shown in the figure and state whether the limit exists at these points.

(a) \( \lim_{x \to 1^-} f(x) = \)

(b) \( \lim_{x \to 1^+} f(x) = \)

(c) \( \lim_{x \to 2^-} f(x) = \)

(d) \( \lim_{x \to 2^+} f(x) = \)

(e) \( \lim_{x \to 3^-} f(x) = \)

(f) \( \lim_{x \to 3^+} f(x) = \)

(g) Does the limit exist at \( c = 1 \)? Yes/No

(h) Does the limit exist at \( c = 2 \)? Yes/No

(i) Does the limit exist at \( c = 3 \)? Yes/No

Problem 5. Given \( \lim_{x \to 2} f(x) = 5 \) and \( \lim_{x \to 2} g(x) = 2 \), use limit laws to compute the following limits or explain why we cannot find the limit. Make sure to keep the \( \lim_{x \to a} \) operator until the very last step.

1. \( \lim_{x \to 2} (2f(x) - g(x)) \)

2. \( \lim_{x \to 2} (f(x)g(2)) \)

3. \( \lim_{x \to 2} \frac{f(x)g(x)}{x} \)

4. \( \lim_{x \to 2} f(x)^2 + x \cdot g(x)^2 \)

5. \( \lim_{x \to 2} [f(x)]^2 \)

6. \( \lim_{x \to 2} \frac{f(x) - 5}{g(x) - 2} \)

Problem 6. Let \( f(x) = \begin{cases} 3bx^2 & \text{if } x < 1/2 \\ 3c(1-x)^2 & \text{if } x \geq 1/2 \end{cases} \). Find \( b \) and \( c \) so that \( f \) is continuous and \( f(1/2) = 1 \). Justify your answer.

Problem 7. The theory of relativity tells us that the length observed by an observer in relative motion with respect to the object is given by

\[
L = L_0 \sqrt{1 - v^2/c^2}
\]

where \( L_0 \) is the length of the object in its rest frame. What happens to \( L \) as \( v \to c^- \)?

Problem 8. In the theory of relativity, the mass of a particle with velocity \( v \) is:

\[
m = \frac{m_0}{\sqrt{1 - v^2/c^2}}
\]

where \( m_0 \) is the mass of the particle at rest and \( c \) is the speed of light. What happens as \( v \to c^- \)?
Problem 9. Physicists have observed that Einstein’s theory of **special relativity** reduces to Newtonian mechanics in the limit as $c \to \infty$, where $c$ is the speed of light. This is illustrated by a stone tossed up vertically from ground level so that it returns to Earth 1 s later. Using Newton’s Laws, we find that the stone’s maximum height is $h = g/8$ meters ($g = 9.8 \text{m/s}^2$). According to special relativity, the stone’s mass depends on its velocity divided by $c$, and the maximum height is

$$h(c) = c \sqrt{\frac{c^2}{g^2} + \frac{1}{4} - \frac{c^2}{g}}$$

Determine $\lim_{c \to \infty} h(c)$.

Problem 10. For each limit, evaluate the limit or explain why it does not exist.

1. $\lim_{x \to 2} \frac{x - 2}{x^2 - 4}$
2. $\lim_{x \to 2} \frac{x - 2}{\frac{1}{x} - \frac{1}{2}}$
3. $\lim_{x \to 0^+} \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x^2 + x}}$
4. $\lim_{t \to 0} \left( \frac{e^{2t} + 4e^t - 5}{e^t - 1} \right)$
5. $\lim_{x \to 4} \frac{x^2 - 16}{\sqrt{x} - 2}$
6. $\lim_{x \to 0} \frac{(2 + x)^2 - 4}{x}$

The following identity may be useful for the next problems.

$$\cos(x + y) = \cos(x) \cos(y) - \sin(x) \sin(y) \quad (1)$$

Problem 11. Use equation (1), to simplify and compute the limit

$$\lim_{h \to 0} \frac{\cos(x + h) - \cos(x)}{h}$$

Problem 12. Evaluate the following limits

1. $\lim_{t \to 0} \frac{\sin^{10}(t)}{t^{10}}$
2. $\lim_{t \to \pi/4} \frac{\sin(t)}{t}$
3. $\lim_{x \to 0} \frac{\tan(x)}{x}$
4. $\lim_{x \to 0} \frac{1 - \cos(2x)}{\sin(2x)}$
5. $\lim_{h \to 0} \frac{\sin(2h)(1 - \cos(h))}{h^2}$
6. $\lim_{x \to 0} \frac{\cos x - \cos(4x)}{x^2} \quad \text{Hint: }$ Rewrite $\cos(4x)$ as $\cos(x + 3x)$ and use (1).
Problem 13. Let $g$ be a function such that, for all real numbers $x$ near 5 but not equal to 5:

$$4 \cos(x - 5) \leq g(x) \leq \frac{1}{5}x + \frac{5}{x} + 2.$$  

Argue that $\lim_{x \to 5} g(x)$ exists and find its value. As usual, justify your answer.

Problem 14.

What does the Squeeze Theorem say about $\lim_{x \to 7} f(x)$ if $\lim_{x \to 7} l(x) = \lim_{x \to 7} u(x) = 6$ and $f(x), u(x),$ and $l(x)$ are related as in the figure to the right? The inequality $f(x) \leq u(x)$ is not satisfied for all $x$. Does this affect the validity of your conclusion?

Problem 15. Find the horizontal asymptote(s) of $f(x) = \frac{2e^x + 3e^{2x}}{e^{2x} + e^{3x}}$.

Problem 16. Evaluate the following limits

1. $\lim_{x \to \infty} \frac{x + 2}{x^2 - 4}$
2. $\lim_{x \to -\infty} \frac{x - 2}{x^2 - 4}$
3. $\lim_{x \to \infty} \frac{x - 2}{\sqrt{x^2 + 4}}$
4. $\lim_{t \to -\infty} \frac{t - 2}{\sqrt{t^2 + 4}}$
5. $\lim_{t \to -\infty} \frac{t^3 + 9t}{10t + 3}$
6. $\lim_{t \to \infty} \frac{t^{4/3} + t^{1/3}}{(2t^{2/3} + 2)^2}$

Problem 17. The hill function is given by

$$H(S) = \frac{S^n}{K^n + S^n}$$

where $n$ and $K$ are positive constants. Find $\lim_{S \to \infty} H(S)$.

Problem 18. The Lennard-Jones potential has the following form,

$$U(r) = \frac{B}{r^{12}} - \frac{A}{r^6}$$

where $A$ and $B$ are constants and $r$ is the distance between the particle in question. Compute

$$\lim_{r \to \infty} U(r).$$
Problem 19. Let \( f(x) = \frac{e^x}{e^x - 2} \). Show that \( f(0) < 1 < f(\ln(4)) \).

Can you use the intermediate value theorem to conclude that there is a solution of \( f(x) = 1 \)?

Can you find a solution to \( f(x) = 1 \)?

Problem 20.

a. Explain in detail how you would use the intermediate value theorem to show that the equation \( xe^x = 2 \) has a solution in the interval \((0, 1)\). (Note: \( e \approx 2.7 \)) Determine if the solution lies in the interval \((0, 1/2)\) or \((1/2, 1)\).

b. Suppose that \( h \) is a continuous function and has the following values

<table>
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<th>( t )</th>
<th>-1</th>
<th>1</th>
<th>-0.1</th>
<th>0.1</th>
<th>-0.01</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h(t) )</td>
<td>2</td>
<td>.5</td>
<td>3</td>
<td>2</td>
<td>-0.3</td>
<td>-2</td>
</tr>
</tbody>
</table>

List all of the smallest intervals for which \( h \) must have a root (zero).

Problem 21. Given \( f(x) = \frac{1}{3x + 1} \), find \( f'(1) \) using the limit definition of the derivative.

Problem 22. Use the definition of the derivative to find the derivative of \( f(x) = \sqrt{2x + 5} \) at \( x = 2 \).