Solutions Manual

Selected odd-numbers problems from

Chapter 3

of

PROOF:
INTRODUCTION TO HIGHER MATHEMATICS

Seventh Edition

Warren W. Esty and Norah C. Esty

5 4 3 2 1

Chapter 3: Proofs

Section 3.1. Inequalities

A1. $x > -2$ A3. $x > 1$
A5. All of them (4, 5, and 7)
A7. all but A, I, J.
B, C, D, E, F, (G already is), H (unnecessary, but true), not I and not J.
A9. If $x < z$, then $4x < 4z$ by Theorem 13A. Then $4x+10 < 4z+10$ by T4.
A11. $x < z \Rightarrow -2x > -2z$ [T13B] $\Rightarrow 5 - 2x > 5 - 2z$ [T4, adding 5]

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B3. $c = 0$, $a = 1$, $b = 3$
B7. Use 8F and 8G.

Section 3.2. Absolute Values

^^^ **The given counterexample is not the only possible counterexample**.
A1. F. $x = -9$ [any $x \leq -7$] A3. F. $x = 7$ [any $x \geq 5$]
A5. F, $x = -4$ A7. T
A9. F. $x = -30$ [any $x \leq -20$] A11. F. $a = -3$, $b = -2$
A13. F. $a = 2$, $b = -3$
A15. F. $a = -2$, $b = -3$
A17. T
A19. F. $c = -1$, $a = 1$
A21. F. $a = -3$, $b = 3$

A23. $2 < x < 4$ A25. $-5 < x < -3$ A27. $6 < x < 8$
A29. $1.45 < x < 1.55$ A31. $a - \delta < x < a + \delta$
A33. $|x - 2.35| < .15$ A35. $|x - 5.715| < .015$

A37. False, $b = -3$, $c = 2$.
A39. False. $a = 1$ and $b = 2$.
A41. False. $a = 3$, $b = -5$, $c = 5$.
A43. False. $a = 1$, $b = 2$, $c = -2$.

A45. Yes, Theorem 5 would be true.
A47. $|x - p| < k$ is $p - k < x < p + k$.

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B1. $c \geq 0$ or $c < 0$. [Proof by Cases]
$x \geq 0$ or $x < 0$. If c or $x = 0$, both sides are 0 and it is true.
If $x = 0$, $cx = 0$ and both sides are 0, so it is true. So, consider $x > 0$. [continue from here]
Another case: $x < 0$ and $c < 0$. One possible proof: Then $cx > 0$ [T3.1.8D] and $|cx| = cx$ (D1)
$= (-c)(-x)$ [algebra] $= |-c||-x|$ [D1] $= |c||x|$ [T2B]
[The rest is omitted here. Do it yourself.]

B3. If $x < 0$, then $|x| = -x$ [by D1] > 0 [by 3.1.8C] $\geq x$ [because $x < 0$ and transitivity]
Multiplying by -1, $-|x| = x \leq x$, which yields the left side.

Section 3.3. Theory of Proofs

A1. Theorems often use a letter to represent an infinite number of cases, and they often can be proved using that letter to represent all cases simultaneously in what is called a *representative-case proof*.
A3. A sentence is a tautology iff its form is always true.
A statement that is always true because of the arrangement of the connectives.
A5. A sentence is a contradiction iff its form is always false.
A statement that is always false because of the arrangement of the connectives.
A7. Give an example of an x such that $H(x)$ is true and $C(x)$ is false.

A9. only (d)
A11. Yes A13. No
A15. No. A17. Yes

A19. Y A21. Y A23. Y A25. N A23. Y A25. N

A27. Proof by exhaustion
A29. "$H \Rightarrow C$ iff (not C) $\Rightarrow$ (not H)"
A31. C and A and $B \Rightarrow D$
A33. C and $(A \Rightarrow B) \Rightarrow D$

A35. $(A \Rightarrow B)$ and $(B \Rightarrow C)$ and $(C \Rightarrow D) \Rightarrow (A \Rightarrow D)$
A37. F, $x = -9$ A39. F, $x = 7$
A41. F, $x = -4$ A43. T
A45. F, $x = -30$ A47. F, $a = -3$, $b = -2$ A48. F, $a = 2$, $b = -3$

A49. F, $a = 2$, $b = -3$ A51. F, $a = -2$, $b = -3$ A53. T
A55. F, $c = -1$, $a = 1$ A57. F. $a = -3$, $b = 3$

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B15. Conjectures 1, 3, 5, 7 follow logically
B17. C1, 7 follow logically

B19. Let x in (0, 4). Choose $y = (x + 4)/2$. [Claim] Then y is in (0, 4) and $y > x$.
B21. $|x - L| < L/2$ implies $-L/2 < x - L$ implies $L/2 < x = |x|$.

B23. True. $A \wedge B \Rightarrow A \Rightarrow C$.
B25. True.
B27. True, by Cases.

B29. No. B31. Yes. B33. No.
B35. Yes. B37. Yes. B39. No.

B41. Yes. $A \wedge B \Rightarrow B \Rightarrow C$. $A \wedge C \Rightarrow D$. Thus $C \wedge D$. $C \wedge D \Rightarrow R$ [from the "or" hypothesis].
B43. Yes. $A \Rightarrow B$ or C. If $A \Rightarrow B$, we are done. If $A \Rightarrow C$, then $C \Rightarrow D$.
B45. No. A is T, B is F, D is F, and C is F.

B47. Short way: $A \Rightarrow A$ or B [taut] $\Rightarrow C$ [hyp]
[Done, by hypotheses in the conclusion]
A longer way: D and $E \Rightarrow E$ (tautology). Now sub $A \Rightarrow C$ for D and $B \Rightarrow C$ for E to get
$(A \Rightarrow C)$ and $(B \Rightarrow C) \Rightarrow (A \Rightarrow C)$. Now rewrite the left side using "cases"
$[(A$ or $B) \Rightarrow C] \Rightarrow (A \Rightarrow C)$.

B49. A and $B \Rightarrow B$ [taut] $\Rightarrow C$. [hyp]

B51. A and not(B and C) LE A and ((not B) or (not C))
LE (A and (not B)) or (A and (not C)). b) Yes.

B53. h follows by contrapositive (none of the others follow)

B57. True. [Pick a particular small h ad show it works

B59. True. Choose $h = 1/3$ [or less]. Then $5 < x < 5\ 1/3$ implies $3x < 16$.

^^^^ **Calculus**

B61. Let $c > 0$. Choose $d = c/4$. Then $|f(x) - 10| = |4x - 2 - 10| = 4|x - 3| < 4(c/4) = c$.

B63. Let $c > 0$. Choose $d = c/3$ [or less]. Then [continue]

B65. Let $\varepsilon > 0$. Choose $b = (1/\varepsilon)^2$. [or less]. Then $x > b = (1/\varepsilon)^2$ implies $1/x < \varepsilon^2$
and $1/\sqrt{x} < \varepsilon$.

B67. Let $k > 0$. Choose $b = e^k$. Then $x > b = e^k$ implies $\ln(x) > k$.

^^^^ **Continuous Functions**

[abbreviated]

B69. Let $\varepsilon > 0$. Choose $\delta = \varepsilon/3$. ...

B71. Let $\varepsilon > 0$. Choose $\delta = \varepsilon/2$. ...

B73. Choose $\varepsilon = 1/2$. Then, for any $\delta > 0$, there exists n so large that $1/(\pi/2 + 2n\pi) < \delta$. Then let $x = 1/(\pi/2 + 2n\pi)$. $\sin(\pi/2 + 2n\pi) = 1$, so $\text{sim}(1/x) = 1 > 1/2$.

B75. "There exists $\varepsilon > 0$ such that for all $\delta > 0$ there exists x such that $|x - 3| < \delta$ and $|f(x) - 10| \geq \varepsilon$." Choose $\varepsilon = 1$. For $\delta > 0$, let $x = 3 + \delta/2$. Then $|x - 3| < \delta$ and $f(x) = 2x + 5 > 2(3)+5 = 11$, so $|f(x) - 10| > 1$.

B77. Let $\varepsilon > 0$. Choose $m = 1/\varepsilon$

^^^^ **Other**

B83. n is divisible by 4, so $n = 4k$ for some k. [continue]

Section 3.4. Proof by Contradiction or Contrapositive

A1. See Definition 3.

A3. If $x \leq 5$ and $y \leq 5$, then $x + y \leq 10$ (by T3.1.12)

A5. If $x \geq 0$ and $y \geq 0$, then $x+y \geq 0$ and $|x+y| = x+y = |x|+|y|$.

A7. [Use the contrapositive and DeMorgan's Law]

A9. If $a = b$, then $a^2 = b^2$ and $(a^2 + b^2)/2 = a^2 = ab$. [by Contrapositive.]

A11. Form: "A and $B \Rightarrow C$," where A is "n pigeons are in k holes," B is "$k < n$," and C is "At least one hole has at least two pigeons." Proof form: (not C) and $A \Rightarrow$ (not B). This is really proof by a Version of the Contrapositive.

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B3. [Prove the contrapositive.]

Section 3.5. Mathematical Induction

A1. We used $x \geq -1$ so that multiplying both sides by $1+x$ was not multiplying by a negative number and therefore preserved the direction of the inequality.
The last line uses $x^2 \geq 0$ for all x, so dropping that term makes the new expression less than or equal to the old expression.

A3. $(1.04)^5 \geq 1 + 5(.04) = 1.2$

A5. $(0.95)^4 \geq 1 + 4(-.05) = .80$

A7. The base case is $n = 3$. $2^3 = 8 > 6 = 2(3)$.
Given $2^n > 2n$, $2^{n+1} = 2(2^n) > 2(2n)$ by IH $= 2n + 2n > 2n + 2 = 2(n+1)$.

A8. The base case $n = 2$ is previously known (The Triangle Inequality).
$|x_1 + x_2 + ... + x_n + x_{n+1}| = |(x_1 + x_2 + ... + x_n) + x_{n+1}| \leq |x_1 + x_2 + ... + x_n| + |x_{n+1}|$ by the Triangle Inequality $\leq |x_1| + |x_2| + ... + |x_n| + |x_{n+1}|$ by the Induction Hypothesis, as desired.

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B7. Let $S(n)$ be $f(n) < 2$. For case 1: $f(1) < 2$ is given.
$f(n+1) = f(n)/2 + 1 < 2/2 + 1$ [by induction hypothesis] = 2.

B9. Let $S(n)$ be "$x_n < 2$." $S(0)$ is true by inspection.
a) $x_{n+1} = \sqrt{(2 + x_n)} < \sqrt{(2 + 2)}$, since $x_n < 2$ (by the induction hypothesis and the fact that $\sqrt{}$ is an increasing function) = 2.
b) $x_{n+1} = \sqrt{(2 + x_n)} > \sqrt{(x_n + x_n)}$ [since $x_n < 2$ by part (a) and $\sqrt{}$ is an increasing function]
$= \sqrt{(2x_n)} > \sqrt{(x_n(x_n))} = x_n$ (since $x_n \geq 0$ since it is a square root).
[Note: (b) follows from this and (a) without another use of induction]

B11. Let $S(n)$ be "$x_n > 4$." $S(0)$ is true by inspection.
a) $x_{n+1} = \sqrt{(12 + x_n)} > \sqrt{(12 + 4)}$ [since $x_n > 4$ and $\sqrt{}$ is increasing] = 4.
b) [Part (b) uses a second induction proof.] Let $S(n)$ be "$x_{n+1} < x_n$."
$x_{n+2} = \sqrt{(12 + x_{n+1})} < \sqrt{(12 + x_n)}$ [by the induction hypthesis and $\sqrt{}$ is increasing] $= x_{n+1}$

^^^^ **Divisible**

Note: The next few follow from Example 3. They also follow from B16, which is a theorem. But these are simple, so they deserve simple proofs.

B13. Let $S(n)$ be $4^n - 1$ is divisible by 3. For $n = 1$, $4^1 - 1 = 3$ which is divisible by 3.
$4^{n+1} - 1 = 4(4^n) - 1 = 3(4^n) + (4^n - 1)$, both terms of which are divisible by 3, by the induction hyp.
[or] $4^{n+1} - 1 = 4(4^n) - 1 - 3(4^n) + 3(4^n) = 4^n - 1 + 3(4^n)$ is divisible by 3 by the IH and the sum of terms divisible by 3 is divisible by 3.
[or] $4^n - 1 = 3k$ for some k. $4^n = 3k + 1$. $4^{n+1} = 4(3k + 1) - 1 = 3(4k + 1)$

Section 3.6. Bad Proofs

A1. *Proof* is defined in Section 3.1, the first paragraph.

A3. $x = -2$, $c = 1$.

A7. False. (They must be not only true, but prior) A9. True.

A11. False. (The steps might all be correct but the logic faulty, or a step might be true but not prior.)

A13. $x = -5$ is a counterexample. A15. True. (The prior results might be different.)

A17. $a = -1$ and $b = -1$ is a counterexample.

A19. a) yes b) yes c) no d) yes