1. Consider the region $\mathcal{P}$ given below.

(a) (1 credit ___) Describe the region $\mathcal{P}$ in polar coordinates using mathematically correct notation.

$$
\mathcal{P} = \{ (r, \theta) \mid 0 \leq r \leq 1, \frac{\pi}{4} \leq \theta \leq \frac{7\pi}{4} \}
$$

(b) (1 credit ___) Calculate $\iiint_D y \, dA$

$$
\iiint_D y \, dA = \int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{4}} (r \sin \theta) \cdot r \, dr \, d\theta
$$

$$
= \int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{4}} r^2 \sin \theta \, dr \, d\theta
$$

$$
= \left[ \frac{r^3}{3} \sin \theta \right]_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{4}} \frac{r^2}{2} \, dr
$$

$$
= -\cos \theta \left( \frac{r^3}{3} \right) \bigg|_0^{\frac{\pi}{4}}
$$

$$
= -\cos \frac{\pi}{4} \left( \frac{1}{3} \right)
$$

$$
= -\left( \frac{\sqrt{2}}{2} \right) \cdot \frac{1}{3} = 0.
$$
2. Consider the integral \( \int_0^2 \int_{y^2}^4 \sqrt{1 + x^{3/2}} \, dx \, dy \).

(a) (1 credit ___) Sketch the region of integration.

(b) (2 credit ___) Reverse the order of integration and compute the integral.

\[
\int_0^4 \int_0^{\sqrt{x}} \sqrt{1 + x^{3/2}} \, dy \, dx
\]

\[
= \int_0^4 \sqrt{x} \sqrt{1 + x^{3/2}} \, dx
\]

\[
= \frac{2}{3} \left[ u^{3/2} \right]_0^9
\]

\[
= \frac{2}{3} \left( 9 - 1 \right)
\]

\[
= \frac{2}{3} \cdot 8 = \frac{16}{3}
\]

\[
\text{Question: } 2 \quad \text{Total: } 3
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\[
\text{Credit Points Earned: } \frac{104}{9}
\]
3. (1 credit ____) Convert to polar coordinates to evaluate

\[ \int_{0}^{1} \int_{\sqrt{2} - y^2}^{0} \frac{1}{\sqrt{x^2 + y^2}} \, dz \, dy \quad = \quad V \]

\[ V = \int_{0}^{\pi/3} \int_{0}^{r} \frac{1}{\sqrt{r^2}} \, r \, dr \, d\theta = \int_{0}^{\pi/3} d\theta \int_{0}^{3} \frac{r}{r} \, dr \]

\[ = \pi \left. -r \right|_{0}^{3} \]

\[ = 3\pi \]

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4. Consider the triple integral
\[ \int_0^1 \int_0^{\sqrt[3]{y^2}} \int_0^{xy} dz \, dx \, dy \]
representing a solid \( S \). Let \( R \) be the projection of \( S \) onto the plane \( z = 0 \).

(a) (1 credit) Draw the region \( R \).

(b) (1 credit) Rewrite this integral as a triple integral in the order \( dz \, dy \, dx \). Do not compute the resulting integral.

\[ \int_0^1 \int_0^{\sqrt[3]{y^2}} \int_0^{xy} dz \, dx \, dy \]

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5. (2 credit) A solid object occupies the region inside the cone \( z = \sqrt{x^2 + y^2} \) (that is \( z \geq \sqrt{x^2 + y^2} \)) and between the two spheres \( x^2 + y^2 + z^2 = 4 \) and \( x^2 + y^2 + z^2 = 9 \). Rewrite "BUT DO NOT EVALUATE" the triple integral in the spherical coordinate system.

\[
\iiint_E e^{\frac{1}{2}(x^2 + y^2 + z^2)} \, dV
\]

\[
2 \leq \rho \leq 3 \\
0 \leq \phi \leq \frac{\pi}{4} \\
0 \leq \theta \leq 2\pi
\]

\[
p \cos \phi = \sqrt{(\rho \sin \phi \cos \theta)^2 + (\rho \sin \phi \sin \theta)^2} \\
p \cos \phi = \rho \sqrt{\sin^2 \phi (\cos^2 \theta + \sin^2 \theta)} \\
\cos \phi = \sin \phi \\
\phi = \frac{\pi}{4}.
\]

\[
\iiint_E e^{\frac{1}{2}(x^2 + y^2 + z^2)} \, dV = \iiint_0^3 e^{\frac{1}{2}(\rho^2)} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta
\]
6. (2 credit ____) Use Green's Theorem to evaluate \( \int_C (2xy - y^2) \, dx + x^2 \, dy \) where \( C \) is the boundary of the region enclosed by \( y = x + 1 \) and \( y = x^2 + 1 \), traversed in a counterclockwise manner.

\[
P = 2xy - y^2 \quad P_y = 2x - 2y
\]

\[
Q = x^2 \quad Q_x = 2x
\]

\[
Q_x - P_y = 2x - (2x - 2y) = 2y.
\]

\[
\oint_C P \, dx + Q \, dy = \iint_D Q_x - P_y \, dA
\]

\[
= \iint_D 2y \, dA
\]

\[
D = \{ (x, y) \mid 1 \leq x \leq 1, x^2 + 1 \leq y \leq x + 1 \}
\]

\[
\rightarrow = \int_1^x \int_{x^2 + 1}^{x+1} 2y \, dy \, dx
\]

\[
= \int_1^x \left. y^2 \right|_{x^2 + 1}^{x+1} \, dx
= \int_1^x (x+1)^2 - (x^2 + 1)^2 \, dx
\]

\[
= \int_0^x x^2 + 2x + 1 - (x^4 + 2x^2 + 1) \, dx = \int_0^x -x^4 + x^2 + 2x \, dx.
\]

\[
= \left[ -\frac{x^5}{5} - \frac{x^3}{3} + \frac{x^2}{2} \right]_0^1 = 1 - \frac{1}{5} - \frac{1}{3} = \frac{4}{15}.
\]
7. Let \( F(x, y, z) = < 2x - y, z - x, y + 1 > \).
   (a) (1 credit \( \_ \_ \_ \_ \)) Show that \( F \) is conservative. Justify your answer.

\[
\nabla \times F = \begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
2x & 3y & 2z \\
2x - y & z - x & y + 1
\end{vmatrix}
= < 1, -1, -(0 - 0), -1 - (-1) >
= 0
\]

(b) (2 credits \( \_ \_ \_ \_ \)) Find a function \( f \) so that \( \nabla f = F \).

\[
f_x = 2x - y \quad f = x^2 - yx + g(y, z)
\]

\[
f_y = z - x \quad f = 2y - xy + h(x, z)
\]

\[
f_z = y + 1 \quad f = y^2 + z + i(x, y)
\]

\[
x^2 - yx + g(y, z) = 2y - xy + h(x, z) = y^2 + z + i(x, y)
\]

\[
f(x, y, z) = x^2 + z + 2y - xy.
\]

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8. For each part of problem a-d below, let C be the straight line segment from (1, 0, 1) to (0, 3, 6).

(a) (1 credit ___) Give a parametrization for C, the straight line segment from (1, 0, 1) to (0, 3, 6).

\[ \mathbf{r}(t) = (1-t)\langle 1, 0, 1 \rangle + t \langle 0, 3, 6 \rangle \]
\[ = \langle 1-t, 3t, 1+5t \rangle \quad 0 \leq t \leq 1 \]
\[ x(t) = 1-t \quad x'(t) = -1 \]
\[ y(t) = 3t \quad y'(t) = 3 \]
\[ z(t) = 1+5t \quad z'(t) = 5 \]

(b) (1 credit ___) Calculate \[ \int_C y \, dx. \]

\[ \int_C y \, dx = \int_0^1 3t \, (-1) \, dt = -3 \left[ \frac{t^2}{2} \right]_0^1 = -\frac{3}{2} \]

(c) (1 credit ___) Let \( \mathbf{F} \) be the vector field \( \mathbf{F} = \langle xy, y^2, 1 \rangle \), calculate \( \int_C \mathbf{F} \cdot d\mathbf{r} \).

\[ \mathbf{F}(\mathbf{r}(t)) = \langle (1-t)(3t), (3t)^2, 1 \rangle \]
\[ \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = \langle 3t - 3t^2, 9t^2, 1 \rangle \cdot \langle -1, 3, 5 \rangle \]
\[ = 3t^2 - 3t + 27t^2 + 5 \]
\[ \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (30t^2 - 3t + 5) \, dt \]
[\( (10t^3 - \frac{3}{2}t^2 + 5t) \bigg|_0^1 \) = 10 - \frac{3}{2} + 5
\[ = 15 - \frac{3}{2} = \frac{27}{2} \]
(d) (1 credit ___) Calculate \( \int_C xy^2 \, dz \). 

\[ \| r'(t) \| = \sqrt{(t)^2 + 3^2 + 5^2} = \sqrt{35} \]

\[ \int_C xy^2 \, dz = \int_0^1 (1-t)(3t^2+5t) \sqrt{35} \, dt \]

\[ = \frac{9}{\sqrt{35}} \int_0^1 t^2 + 4t^3 - 5t^4 \, dt \]

\[ = \frac{9}{\sqrt{35}} \left( \frac{t^3}{3} + t^4 - t^5 \right) \bigg|_0^1 \]

\[ = \frac{9}{\sqrt{35}} \left( \frac{1}{3} \right) = \frac{3}{\sqrt{35}} \]

9. (1 credit ___) Compute \( \int_C \mathbf{F} \cdot d\mathbf{r} \) where \( \mathbf{F}(x,y) = \langle (1+xy)e^{xy}, e^y + x^2 e^y \rangle \) and \( C \) is as pictured. Note that if \( f(x,y) = e^y + xe^{xy} \), then \( \nabla f = \mathbf{F} \).

\[ f(0,0) = e^0 + 0 = 1 \]

\[ f(1,1) = e^1 + e^1 = 2e \]

\[ \int_C \mathbf{F} \cdot d\mathbf{r} = f(1,1) - f(0,0) = 2e - 1 \]