M273Q Multivariable Calculus   Spring 2017   Review Problems for Exam 1

1. True or False? Circle ONE answer for each. *Hint: For effective study, explain why if 'true' and give a counterexample if 'false.'*
   (a) ____ T or F   There exists a function f with continuous second-order partial derivatives such that $f_x(x,y) = x + y^2$ and $f_y(x,y) = x - y^2$.
   (b) ____ T or F   If $f(x,y) = \ln y$, then $\nabla f(x,y) = \dfrac{1}{y}$.
   (c) ____ T or F   If $f(x,y) \to L$ as $(x,y) \to (a,b)$ along every straight line through (a,b), then $\lim_{(x,y)\to(a,b)} f(x,y) = L$.
   (d) ____ T or F   If f has a local minimum at (a,b) and f is differentiable at (a,b), then $\nabla f(a,b) = \mathbf{0}$.
   (e) ____ T or F   If $f(x,y) = \sin x + \sin y$, then $-\sqrt{2} \le D_{\mathbf{u}} f(x,y) \le \sqrt{2}$.

2. Find and sketch the domain of the function $f(x,y) = \sqrt{4 - x^2 - y^2} + \sqrt{1 - x^2}$.

3. Sketch several level curves of the function $v(x,y) = e^x + y$.

4. Consider the function $f(x,y) = \dfrac{1}{x^2 + y^2 + 1}$.
   (a) Find equations for the following level curves for f, and sketch them.
      (a) $f(x,y) = \frac{1}{5}$
      (b) $f(x,y) = \frac{1}{10}$
   (c) Find k such that the level curve $f(x,y) = k$ consists of a single point.
   (d) Why is k the global maximum of $f(x,y)$?

5. Evaluate the limit or show that it does not exist. (There will NOT be $\epsilon - \delta$ proofs on the exam).
   (a) $\displaystyle\lim_{(x,y)\to(1,1)} \frac{2xy}{x^2 + 2y^2}$
   (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2 + 2y^2}$

6. Find the first partial derivatives.
   (a) $u = e^{-r}\sin(2\theta)$
   (b) $g(u,v) = u\tan^{-1} v$

7. Find all second partial derivatives.
   (a) $z = xe^{-2y}$
   (b) $v = r\cos(s + 2t)$

8. If $z = y^2 e^x, x = \cos t, y = t^3$, find $\dfrac{dz}{dt}$.

9. If $z(x,y) = x\sin y, x(s,t) = se^t, y(s,t) = se^{-t}$, find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$.

10. Suppose $z = e^r\cos\theta, r = st$, and $\theta = \sqrt{s^2 + t^2}$.
    (a) State the chain rule for $\dfrac{\partial z}{\partial s}$.
    (b) Find $\dfrac{\partial z}{\partial s}$ in terms of s and t only.

11. Let $f(x,y,z) = xze^{x+y^2}$.
    (a) Find $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial f}{\partial z}$.
    (b) Find $\displaystyle\lim_{(x,y,z)\to(-1,1,1)} f(x,y,z)$.
    (c) Find $\nabla f(-1,1,1)$.
    (d) Find the directional derivative of f at $(-1,1,1)$ in the direction of $\mathbf{v} =< 1, 2, -1 >$.

(e) Approximate the greatest increase in f from moving 0.01 units in any direction from $(-1, 1, 1)$.

12. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

   (a) $xy + yz - xz = 0$

   (b) $\ln(x + yz) = 1 + xy^2z^3$.

13. Find an equation of the tangent plane to the given surface at the specified point.

   (a) $z = e^x \cos y$, $(0, 0, 1)$

   (b) $z = xe^{\sin y}$ at $(2, \pi, 2)$.

   (c) $x^2z(2y + z)^2 = 4$ at $(2, -1, 1)$.

14. Use an appropriate tangent plane to approximate $(0.999)^7(1 + 2\sin(0.02))$.

15. The temperature distribution of a ball centered at the origin is given by $T(x, y, z) = \dfrac{25}{x^2 + y^2 + z^2 + 1}$. Find the maximum rate of increase in temperature at $(3, -1, 2)$ and find a unit vector in that direction.

16. If $v = x^2 \sin y + ye^{xy}$, where $x = s + 2t$ and $y = st$, use the Chain Rule to find $\dfrac{\partial v}{\partial s}$ and $\dfrac{\partial v}{\partial t}$ when $s = 0$ and $t = 1$.

17. Find the direction in which $f(x, y, z) = ze^{xy}$ increases most rapidly at the point $(0, 1, 2)$. What is the maximum rate of increase?

18. Find the points on $z^2 = x^2 + y^2$ that are closest to $(2, 2, 0)$.

19. Locate all relative maxima, minima, and saddle points for $f(x, y) = x^3 + y^2 - 12x + 6y - 7$.

20. Let $f(x, y, z) = \sqrt{x^2 - yz}$.

   (a) Find a *unit* vector that points in the direction in which f increases most rapidly at $P(3, 2, 4)$.

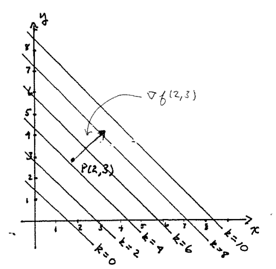
   (b) What is the rate of change of f at $P(3, 2, 4)$ in the direction found in a.

   (c) Find an equation of the tangent plane to $\sqrt{x^2 - yz} = 1$ at $P(3, 2, 4)$.

   (d) Given $\sqrt{x^2 - yz} = 1$, find $\dfrac{\partial z}{\partial y}$ at $P(3, 2, 4)$.

   (e) Without using a calculator, give a good linear approximation of $\sqrt{(3.1)^2 - (1.9)(4.2)}$

21. The picture below is a contour (level curve) plot of a function $z = f(x, y)$ of two variables. Assume that the distance between adjacent drawn curves is 1 unit.



   (a) Sketch $\nabla f(2, 3)$ with appropriate direction and length.

   (b) Using part a, estimate the rate of change of f at $P(2, 3)$ in the direction of $< 3, 4 >$.

   (c) Suppose an object moves across $P(2, 3)$ with velocity $< 3, 4 >$. Using part b, estimate the time rate of change of f.

22. Find all critical points of $f(x, y) = x^2 + 4xy + y^2 - 2x + 8y + 3$ and classify each as being a point at which f has a local (relative) max, min, or saddle.

23. Find the max and min of $f(x, y) = 2x^2 + y^2 - 2x$ subject to $x^2 + y^2 = 4$. What are the absolute max and absolute min of $f(x, y) = 2x^2 + y^2 - 2x$ on the region $x^2 + y^2 \leq 4$?

24. Let $f(x, y) = 4 - (x - 1)(y - 1)$ with $D = \{(x, y)|0 \leq y \leq 4 - x^2.\}$

   (a) Find and classify critical points of f with the second derivative test.

(b) Is D closed and bounded? What points on the boundary $y = 0$ could potentially be absolute maxima or minima?

(c) Write the upper boundary of D as a constraint and use Lagrange multipliers to find critical points subject to this constraint.

(d) What are the absolute max and min of f on D?

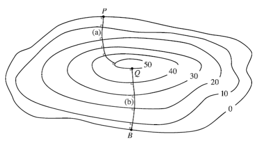
25. Find the local maximum and minimum values and saddle points of the function $f(x, y) = x^3 - 6xy + 8y^3$.

26. Find the absolute maximum and minimum values of $f(x, y) = e^{-x^2-y^2}(x^2 + 2y^2)$ on D where D is the disk $x^2 + y^2 \leq 4$.

27. Use Lagrange multipliers to find the maximum and minimum values of $f(x, y) = \dfrac{1}{x} + \dfrac{1}{y}$ subject to the constraint $\dfrac{1}{x^2} + \dfrac{1}{y^2} = 1$.

28. Find the points on the surface $xy^2z^3 = 2$ that are closest to the origin.

29. Below is a topographical map of a hill.



(a) Starting at P, sketch the path of steepest ascent to the peak elevation of 50 yards.

(b) Suppose it rains, and water runs down the hill starting at Q. At what point would you expect the water to reach the bottom? Justify your answer.