M273 Exam 1 Overview

This overview is provided to you as a brief listing of relevant formulas and information you'll likely need on the exam. This list is not exhaustive. You may or may not need everything on this list to succeed on the exam.

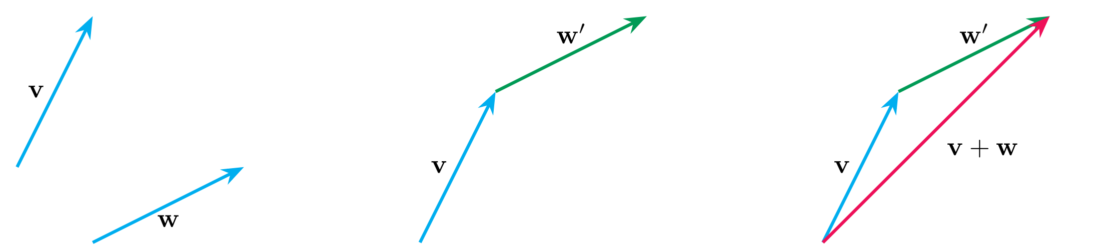
The formulas in the box will be the only ones provided on the exam

$$\kappa(s) = \left|\left|\frac{d\mathbf{T}}{ds}\right|\right| \qquad \kappa(x) = \frac{|f''(x)|}{[1+(f'(x))^2]^{3/2}}$$

$$\kappa(t) = \frac{||\mathbf{T}'(t)||}{||\mathbf{r}'(t)||} \qquad \kappa(t) = \frac{||\mathbf{r}'(t)\times\mathbf{r}''(t)||}{||\mathbf{r}'(t)||^3}$$

§12.1 Vectors in the Plane

- $\mathbf{v} = \langle v_1, v_2 \rangle$
- Magnitude of a vector: $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2}$
- Geometric vector addition "tip-to-tail"



$\mathbf{v}$ and $\mathbf{w}$ in the plane — $\mathbf{v}$ and $\mathbf{w}'$ a translate of $\mathbf{w}$ — $\mathbf{v}$ and $\mathbf{v}+\mathbf{w}$ with same tail

- $\mathbf{e_v} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ a unit vector which points in the direction of $\mathbf{v}$

§12.2 Vectors in Three Dimensions

- $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$
- Parameterization of a *line* through the point (x_0, y_0, z_0) with direction vector $\mathbf{v}$

$$\mathbf{r}(t) = \langle x_0, y_0, z_0 \rangle + t\mathbf{v} \qquad \text{for} \qquad t \in \mathbb{R}$$

- Parameterization of a *line segment* through the point (x_0, y_0, z_0) with direction vector $\mathbf{v}$

$$\mathbf{r}(t) = \langle x_0, y_0, z_0 \rangle + t\mathbf{v} \qquad \text{for} \qquad t \in [a, b]$$

- Parameterization of the *line segment* from point P to point Q using a *convex combination*

$$\mathbf{r}(t) = \overrightarrow{OP}(1-t) + \overrightarrow{OQ}t \qquad \text{for} \qquad t \in [0, 1]$$

§12.3 Dot Product and the Angle Between Two Vectors

- Dot product between two vectors $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$$

- $\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \|\mathbf{w}\| \cos\theta$ where $0 \leq \theta \leq \pi$ is the angle between the two vectors
- If $\mathbf{v} \cdot \mathbf{w} = 0$ then $\mathbf{v}$ and $\mathbf{w}$ are orthogonal
- $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$
- Projection: given two vectors $\mathbf{u}$ and $\mathbf{v} \neq \mathbf{0}$, then the projection of $\mathbf{u}$ onto $\mathbf{v}$ is given by

$$\mathbf{u}_{||\mathbf{v}} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$

- Orthogonal decomposition: given two vectors $\mathbf{u}$ and $\mathbf{v} \neq \mathbf{0}$ then $\mathbf{u}$ can be written as the sum of two orthogonal vectors

$$\mathbf{u} = \mathbf{u}_{||\mathbf{v}} + \mathbf{u}_{\perp\mathbf{v}}$$

§12.4 The Cross Product

- Given two vectors $\mathbf{v}$ and $\mathbf{w}$ the cross product is the vector

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \hat{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \hat{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \hat{k}$$

- $\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\| \|\mathbf{w}\| \sin\theta$
- Anticommutative: $\mathbf{w} \times \mathbf{v} = -\mathbf{v} \times \mathbf{w}$
- $\mathbf{v} \times \mathbf{w}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$, forming a right-handed system
- The areas of the parallelogram $\mathcal{P}$ and triangle $\mathcal{T}$ spanned by $\mathbf{v}$ and $\mathbf{w}$ is

$$\text{Area}(\mathcal{P}) = \|\mathbf{v} \times \mathbf{w}\| \qquad \text{and} \qquad \text{Area}(\mathcal{T}) = \frac{\|\mathbf{v} \times \mathbf{w}\|}{2}$$

- The volume of the parallelepiped $\mathcal{P}$ spanned by $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ is given by the *scalar triple product*

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = \left| \det \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{pmatrix} \right|$$

§12.5 Planes in 3-Space

- A plane through a point (x_0, y_0, z_0) with normal vector $\mathbf{n} = \langle a, b, c\rangle$ has equation

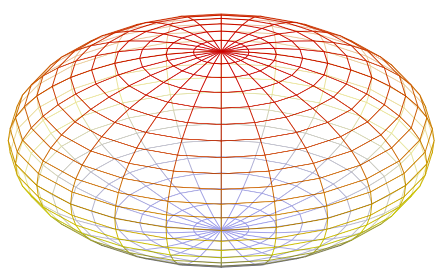
$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

- Parallel planes have parallel normal vectors
- Perpendicular planes have perpendicular normal vectors
- The direction vector of the line of intersection of two planes is parallel to the cross product of the plane's normal vectors

§12.6 A Survey of Quadric Surfaces

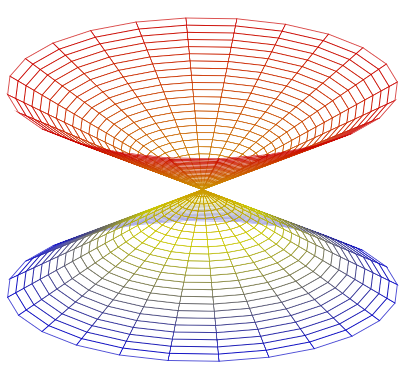
- Ellipsoid

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$$



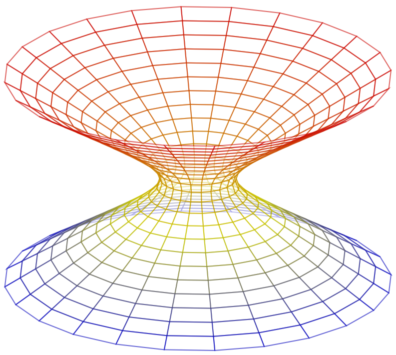
- Elliptic Cone

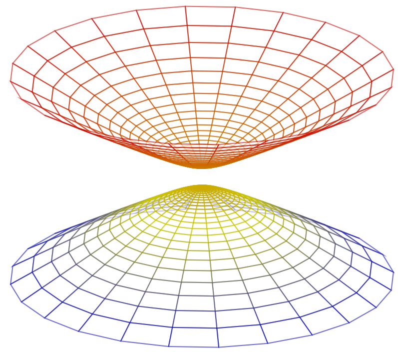
$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \left(\frac{z}{c}\right)^2$$



- Hyperboloids

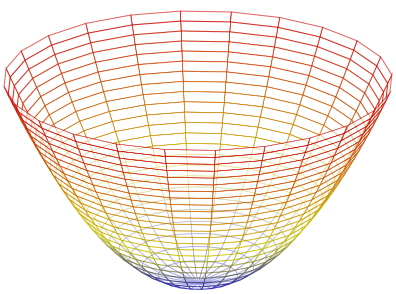
One Sheet: $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \left(\frac{z}{c}\right)^2 + 1$ and Two Sheet: $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \left(\frac{z}{c}\right)^2 - 1$

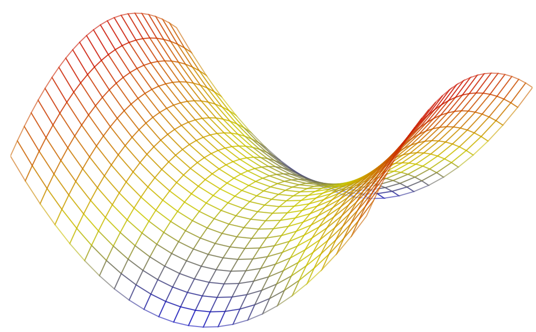




- Paraboliods

Elliptic Paraboloid: $z = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2$ and Hyperbolic Paraboloid: $z = \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2$





§13.1 Vector-Valued Functions

- A vector-valued function is another name for a parameterization
- $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ is a vector-valued function

§13.2 Calculus of Vector-Valued Functions

- You can perform limits, differentiation, and integration componentwise
- Tangent line to $\mathbf{r}(t)$ at $t = t_0$

$$\mathscr{L}(t) = \mathbf{r}(t_0) + t\mathbf{r}'(t_0)$$

§13.3 Arc Length and Speed

- Speed: given a parameterization $\mathbf{r}(t)$ then the speed is given by $||\mathbf{r}'(t)||$
- Arc Length s is the actual distance traveled on a path. The arc length of $\mathbf{r}(t)$ for $t \in [a, b]$ is

$$s = \int_a^b ||\mathbf{r}'(t)||\, dt$$

- An arc length parameterization of a given parameterization $\mathbf{r}(t)$ is a re-parameterization with the parameter s so that the curve is traced at unit speed; to find one
  1. Form the arc length function and evaluate the integral

  $$s = g(t) = \int_a^t ||\mathbf{r}'(u)||\, du$$

  2. Determine the inverse $t = g^{-1}(s)$ (i.e. solve for t)
  3. The arc length parameterization $\mathbf{r}_1(s)$ is found by performing the substitution

  $$\mathbf{r}_1(s) = \mathbf{r}(g^{-1}(s))$$

§13.4 Curvature

- The unit tangent vector $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||}$
- The unit normal vector $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{||\mathbf{T}'(t)||}$
- The unit tangent vector points in the direction of motion
- The unit normal vector points in the direction of bending
- $\mathbf{T}$ and $\mathbf{N}$ are orthogonal
- The binormal vector $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$
- All four curvature formulas will be provided on the exam (see formulas at the top of page one)

§13.5 Motion in 3-Space

- If $\mathbf{r}(t)$ is the position then the velocity is $\mathbf{v}(t) = \mathbf{r}'(t)$ and the acceleration $\mathbf{a}(t) = \mathbf{r}''(t)$
- The acceleration $\mathbf{a}(t)$ has an orthogonal decomposition

$$\mathbf{a}(t) = a_{\mathbf{T}}\mathbf{T}(t) + a_{\mathbf{N}}\mathbf{N}(t)$$

  where $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$ are the tangential and normal scalar components of projection given by

$$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \frac{\mathbf{a} \cdot \mathbf{v}}{\|\mathbf{v}\|} \qquad \text{and} \qquad a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \sqrt{\|\mathbf{a}\|^2 - |a_{\mathbf{T}}|^2}$$

- The above line of formulas will be provided if needed on the exam