M273 Exam 2 Overview

This overview is provided to you as a brief listing of relevant formulas and information you'll likely need on the exam. This list is not exhaustive. You may or may not need everything on this list to succeed on the exam.

§14.1 Functions of Two or More Variables

- A function $f(x, y)$ has a domain which is a subset of $\mathbb{R}^2$
- A vertical trace is obtained by intersecting a surface $z = f(x, y)$ with a plane $x = a$ or $y = b$
- A horizontal trace (or level curve) is obtained by intersecting a surface $z = f(x, y)$ with a plane $z = c$
- A contour plot/map is obtained by considering many horizontal traces of a surface $z = f(x, y)$ and projecting all of them onto the xy-plane

§14.2 Limits and Continuity

- If $f(x, y)$ is continuous at (a, b) then $\lim\limits_{(x,y)\to(a,b)} f(x, y) = f(a, b)$ i.e. "just plug it in"
- Limits can be shown to not exist by considering different paths to (a, b)
- Common paths to try: x-axis, y-axis, $y = mx$, and $y = x^2$
- Unlike in Calc I, if any finite number of paths yield the same limiting value, this does not prove the limit exists and equals this value (*all* paths most agree, nearly impossible to check explicitly)

§14.3 Partial Derivatives

- $f_x(x, y)$ and $f_y(x, y)$ are the first order x- and y-partials of $f(x, y)$, respectively
- f_{xy} is obtained by first differentiating f with respect to x then with respect to y
- $f_{xy} = f_{yx}$ (provided they exist and are continuous)—this generalizes to higher order derivatives

§14.4 Differentiability and Tangent Planes

- Tangent plane to $z = f(x, y)$ at the point (a, b)

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

- Locally to $z = f(a, b)$ the surface can be approximated by the tangent plane, that is

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

§14.5 The Gradient and Directional Derivatives

- $\nabla = \left\langle \dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z} \right\rangle$
- The gradient of a function f

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

- Chain rule for paths: given a function $f(x, y, z)$ and a path parameterized by $\mathbf{r}(t)$ then the rate of change of f along the parameterized path is

$$\frac{d}{dt}f(\mathbf{r}(t)) = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$$

- The rate of change of f in the direction of a unit vector $\mathbf{u}$ is given by

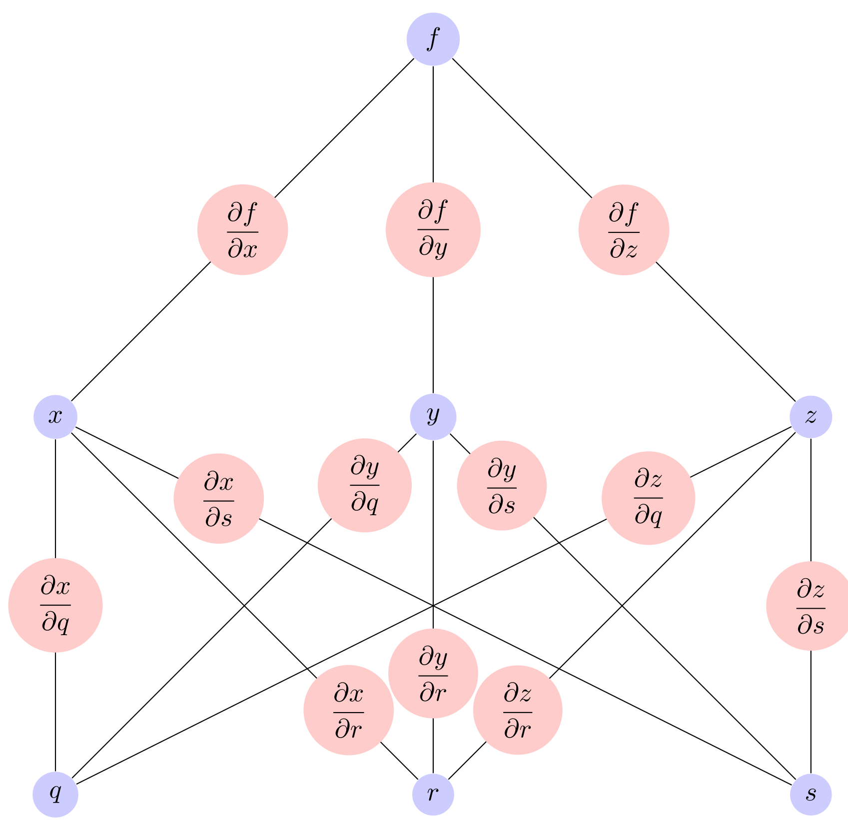
$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}$$

- The gradient always points in the direction of the maximum rate of increase
- The negative gradient always points in the direction of the maximum rate of decrease
- The gradient is always orthogonal to the level curves
- Given an equation of the form $F(x, y, z) = 0$, which is an implicit surface, then the gradient ∇F serves as a normal vector to the surface
- Given the implicit surface $F(x, y, z) = 0$ then the equation of the tangent plane to this surface at the point (a, b, c) is

$$F_x(a, b, c)(x - a) + F_y(a, b, c)(y - b) + F_z(a, b, c)(z - c) = 0$$

§14.6 The Chain Rule

- Given a function $f(x, y, z)$ where $x = x(q, r, s)$, $y = y(q, r, s)$, and $z = z(q, r, s)$ then a dependency tree can be drawn



Then to write down how f changes with respect to q look at each path from f to q and multiply the partials along each connection and add paths together, yielding

$$\frac{\partial f}{\partial q} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial q} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial q} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial q}$$

- Implicit Differentiation: given an equation of the form $F(x, y, z) = 0$ then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \qquad\qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

§14.7 Optimization in Several Variables

- Critical Point: A point (a, b) in the domain of $f(x, y)$ is a critical point if $f_x(a, b) = 0$ or DNE AND $f_y(a, b) = 0$ or DNE
- Discriminant: $D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$
- Second Derivative Test: if (a, b) is a critical point of f and all second order partials are continuous at (a, b) then
  - If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum
  - If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum
  - If $D(a, b) < 0$, then $f(a, b)$ is a saddle
  - If $D(a, b) = 0$, then the test is inconclusive
- Finding absolute extrema of f over a domain $\mathcal{D}$:
  - Find local extrema of f that live in $\mathcal{D}$
  - Restrict f to the boundary of $\mathcal{D}$ by performing some form of a substitution
  - Find local extrema of this restricted f using Calc I methods
  - Compare all candidate values: the biggest is the absolute max and the smallest is the absolute minimum

§14.8 Lagrange Multipliers: Optimizing with a Constraint

- Let f be the function to be optimized (minimized or maximized), this is called the *objective function* subject to the constraint $g(x, y, z) = 0$
- Lagrange Condition:

$$\nabla f = \lambda \nabla g$$

- Find all solutions to the system of equations: $\nabla f = \lambda \nabla g$ and $g(x, y, z) = 0$
- Evaluate f at all solutions to the above system; the highest is the maximum and the lowest is the minimum