Exam 2 Review Problems: Solutions

1) a) False. If one were to exist, by Clairaut's Theorem, its 2^{nd} order mixed partials would have to be equal. Note that:

$$f_{xy} = 2y \quad \text{and} \quad f_{yx} = 1,$$

these are not equal, which contradicts the theorem, therefore the supposition that such a function exist, must be false.

b) False. ∇f is a vector and $\frac{1}{y}$ is a scalar. In particular, $\nabla f = \langle 0, \frac{1}{y} \rangle$.

c) False. There may exist a non-straight line path such that if (a,b) is approached along this path then $f(x,y)$ approaches a different value, or DNE.

d) True. Fermat's Theorem asserts that if f has a local extrema at (a,b) then (a,b) is a critical point. A critical point of f is a point in the domain of f where

1) $f_x(a,b) = 0$ or DNE; and

2) $f_y(a,b) = 0$ or DNE

Since f is differentiable at (a,b) this means $f_x(a,b) = 0$ and $f_y(0,0) = 0$. Which, equivalently, means $\nabla f(a,b) = \langle 0,0 \rangle = \vec{0}$.

e) True. If $f(x,y) = \sin x + \sin y$, then $D_{\vec{u}} f = \nabla f \cdot \vec{u}$ where $\vec{u}$ is a unit vector. Then,

$$\begin{aligned}
\left| D_{\vec{u}} f \right| &= \left| \nabla f \cdot \vec{u} \right| \\
&= \left| \langle \cos x, \cos y \rangle \cdot \vec{u} \right| \\
&= \left| \, \| \langle \cos x, \cos y \rangle \| \, \cancelto{1}{\| \vec{u} \|} \cos\theta \right| \\
&= \left| \sqrt{\cos^2 x + \cos^2 y} \; \cos\theta \right| \\
&\le \left| \sqrt{1+1} \cdot 1 \right|
\end{aligned}$$

$$\left| D_{\vec{u}} f \right| \le \sqrt{2}$$

or equivalently,

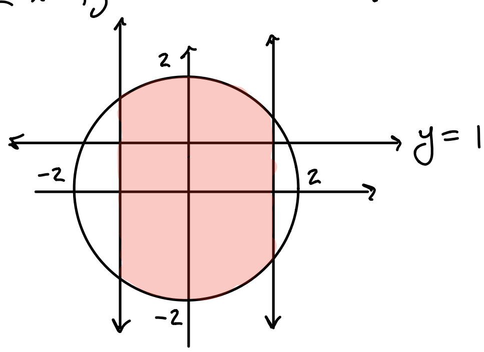
$$-\sqrt{2} \le D_{\vec{u}} f \le \sqrt{2}$$

2) $f(x,y) = \sqrt{4 - x^2 - y^2} + \sqrt{1 - x^2}$

A point (x,y) in the domain of f must simultaneously satisfy:

$4 - x^2 - y^2 \ge 0$ and $1 - x^2 \ge 0$

$4 \ge x^2 + y^2$ $\qquad 1 \ge x^2$



$x^2 \le 1$

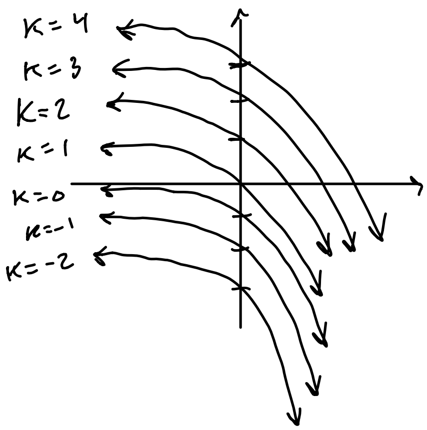
$|x| \le 1$

$\Rightarrow -1 \le x \le 1$

3) $V(x,y) = e^x + y \Rightarrow k = e^x + y$ for $k \in \mathbb{R} \Rightarrow y = k - e^x$

Note:

(1) $\lim\limits_{x \to \infty} (k - e^x) = -\infty$ (2) $\lim\limits_{x \to -\infty} (k - e^x) = k$



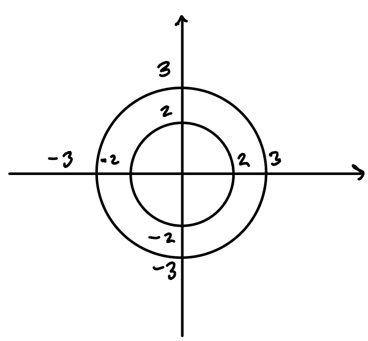
4) $f(x,y) = \dfrac{1}{x^2+y^2+1}$

a) $\dfrac{1}{x^2+y^2+1} = \dfrac{1}{5}$

$x^2 + y^2 = 4$

b) $\dfrac{1}{x^2+y^2+1} = \dfrac{1}{10}$

$x^2+y^2 = 9$



c) $\dfrac{1}{x^2+y^2+1} = k$

$x^2+y^2+1 = \dfrac{1}{k}$

$x^2+y^2 = \dfrac{1}{k} - 1 \Rightarrow \dfrac{1}{k} - 1 = 0$

$k = 1$

d) $K=1$ is the global maximum of $f(x,y)$ since $x^2+y^2+1 \geq 1$ for all $(x,y) \in \mathbb{R}^2$.

5)

a) $\lim\limits_{(x,y)\to(1,1)} \dfrac{2xy}{x^2+2y^2} = \dfrac{2}{3}$

b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{2xy}{x^2+2y^2}$ Approach $(0,0)$ along the line $y=mx$:

$$\lim_{x\to 0} \frac{2x(mx)}{x^2+2(mx)^2} = \lim_{x\to 0} \frac{2m\cancel{x^2}}{\cancel{x^2}(1+2m^2)}$$

$$= \frac{2m}{1+2m^2} \Rightarrow \lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+2y^2}$$ DNE, since the value of the limit is path dependent.

6)

a) $u = e^{-r}\sin(2\theta)$

$u_r = -e^{-r}\sin(2\theta)$

$u_\theta = 2e^{-r}\cos(2\theta)$

b) $g(u,v) = u\tan^{-1}(v)$

$g_u = \tan^{-1}(v)$

$g_v = \dfrac{u}{1+v^2}$

7)

a) $z = xe^{-2y}$

$z_x = e^{-2y}$ $\quad z_y = -2xe^{-2y}$ $\quad z_{xy} = -2e^{-2y}$

$z_{xx} = 0$ $\quad z_{yy} = 4xe^{-2y}$ $\quad z_{yx} = -2e^{-2y}$

$\left.\begin{matrix} z_{xy} = -2e^{-2y} \\ z_{yx} = -2e^{-2y} \end{matrix}\right\}$ Equal.

b) $v = r\cos(s+2t)$

$v_r = \cos(s+2t)$ $\quad v_s = -r\sin(s+2t)$ $\quad v_t = -2r\sin(s+2t)$

$v_{rr} = 0$ $\quad v_{ss} = -r\cos(s+2t)$ $\quad v_{tt} = -4r\cos(s+2t)$

$v_{rs} = -\sin(s+2t) = v_{sr}$ $\quad v_{rt} = -2\sin(s+2t) = v_{tr}$ $\quad v_{st} = -2r\cos(s+2t) = v_{ts}$

8) $z = y^2 e^x$

$x = \cos t$

$y = t^3$

$$\frac{dz}{dt} = \frac{dz}{dx}\frac{dx}{dt} + \frac{dz}{dy}\frac{dy}{dt}$$

$$= y^2 e^x(-\sin t) + 2y e^x (3t^2)$$

$$= -y^2 e^x \sin t + 6y e^x t^2$$

$$\frac{dz}{dt} = -t^6 e^{\cos t}\sin t + 6t^5 e^{\cos t}$$

9) $z = x\sin y$

$x = se^t$

$y = se^{-t}$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$= \sin y\, e^t + x\cos y\, e^{-t}$$

$$\frac{\partial z}{\partial s} = \sin(se^{-t})e^t + s\cos(se^{-t})$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$= \sin y\, se^t + x\cos y(-se^{-t})$$

$$\frac{\partial z}{\partial t} = \sin(se^{-t})se^t - s^2\cos(se^{-t})$$

10) $z = e^r\cos\theta$

$r = st$

$\theta = \sqrt{s^2+t^2}$

a) $$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial s} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial s}$$

b) $$\frac{\partial z}{\partial s} = e^r\cos\theta\, t - e^r\sin\theta\left(\frac{\cancel{2}s}{\cancel{2}\sqrt{s^2+t^2}}\right)$$

$$\frac{\partial z}{\partial s} = e^{st}\cos\left(\sqrt{s^2+t^2}\right)t - e^{st}\sin\left(\sqrt{s^2+t^2}\right)\left(\frac{s}{\sqrt{s^2+t^2}}\right)$$

11) $f(x,y,z) = xze^{x+y^2}$

a) $f_x = ze^{x+y^2} + xze^{x+y^2}$

$f_y = 2xyze^{x+y^2}$

$f_z = xe^{x+y^2}$

b) $\lim_{(x,y,z)\to(-1,1,1)} f(x,y,z) = -e^0 = -1$

c) $\nabla f(-1,1,1) = \langle 0, -2, -1\rangle$

d) $D_{\vec{v}} f = \nabla f \cdot \dfrac{\vec{v}}{\|\vec{v}\|}$

$$= \frac{\langle 0, -2, -1\rangle \cdot \langle 1, 2, -1\rangle}{\sqrt{6}}$$

$$D_{\vec{v}} f = -\frac{3}{\sqrt{6}}$$

e) $\dfrac{\|\nabla f\|}{100} = \dfrac{\|\langle 0, -2, -1\rangle\|}{100}$

$$= \frac{\sqrt{5}}{100}$$

12)

a) $\underbrace{xy + yz - xz}_{:= F} = 0$

$F_x = y - z$

$F_y = x + z$

$F_z = y - x$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{y-z}{y-x}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{x+z}{y-x}$$

b) $\ln(x+yz) = 1 + xy^2z^3$

$\underbrace{\ln(x+yz) - xy^2z^3 - 1}_{:= F} = 0$

$$F_x = \frac{1}{x+yz} - y^2z^3$$

$$F_y = \frac{z}{x+yz} - 2xyz^3$$

$$F_z = \frac{y}{x+yz} - 3xy^2z^2$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{\frac{1}{x+yz} - y^2z^3}{\frac{y}{x+yz} - 3xy^2z^2}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{\frac{z}{x+yz} - 2xyz^3}{\frac{y}{x+yz} - 3xy^2z^2}$$

13)

a) $z = e^x \cos y$; $P = (0, 0, 1)$ $\qquad z = 1 + x$

$\nabla z = \langle e^x \cos y, -e^x \sin y\rangle$

$\nabla z_p = \langle 1, 0\rangle$

b) $z = xe^{\sin y}$; $P = (2, \pi, 2)$ $\qquad$ $\underline{\underline{z = 2 + (x-2) - 2(y - \pi)}}$

$\nabla z = \langle e^{\sin y}, xe^{\sin y}\cos y \rangle$

$\nabla z_p = \langle 1, -2 \rangle$

c) $\underbrace{x^2 z(2y+z)^2}_{:= F} = 4$; $P = (2, -1, 1)$

$\nabla F = \langle 2xz(2y+z)^2, 4x^2 z(2y+z), x^2(2y+z)^2 + 2x^2 z(2y+z) \rangle$

$\nabla F_p = \langle 4, -16, -4 \rangle$

$4(x-2) - 16(y+1) - 4(z-1) = \underline{\underline{0}}$

14) $(0.999)^7(1 + 2\sin(0.02)) \Rightarrow f(x,y) = x^7(1 + 2\sin(y))$; $P = (1,0)$

$\nabla f = \langle 7x^6(1 + 2\sin(y)), 2x^7\cos(y) \rangle$

$\nabla f_p = \langle 7, 2 \rangle \Rightarrow z = f(1,0) + 7(x-1) + 2y$

$z = 1 + 7(x-1) + 2y$

$$(0.999)^7(1 + 2\sin(0.02)) \approx 1 + 7(-0.001) + 0.04$$

$$\approx 1 - 0.007 + 0.04$$

$$\approx 1 + 0.033$$

$$\approx \underline{\underline{1.033}}$$

15) $T(x,y,z) = \dfrac{25}{x^2 + y^2 + z^2 + 1}$; $P = (3, -1, 2)$

$\nabla T = \dfrac{25\langle -2x, -2y, -2z \rangle}{(x^2+y^2+z^2+1)^2}$ $\qquad$ $\nabla T_p = \dfrac{-50\langle 3, -1, 2 \rangle}{(9 + 1 + 4 + 1)^2}$

$\nabla T_p = -\dfrac{50}{225}\langle 3, -1, 2 \rangle$

$$\|\nabla T_p\| = \frac{50\sqrt{14}}{225} = \frac{2\sqrt{14}}{9}$$ & unit vector: $\vec{u} = \left\langle \frac{3}{\sqrt{14}}, -\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}} \right\rangle$

16) $V = x^2 \sin y + y e^{xy}$

$x = s + 2t$

$y = st$

$(s,t) = (0,1)$

$x(0,1) = 2$

$y(0,1) = 0$

$$\left.\frac{\partial V}{\partial x}\right|_{(0,1)} = 2x\sin y + y^2 e^{xy}\Big|_{(0,1)} = 0$$

$$\left.\frac{\partial V}{\partial y}\right|_{(0,1)} = x^2 \cos y + e^{xy} + xy e^{xy}\Big|_{(0,1)} = 4 + 1 = 5$$

$$\left.\frac{\partial x}{\partial s}\right|_{(0,1)} = 1 \qquad \left.\frac{\partial y}{\partial s}\right|_{(0,1)} = t = 1$$

$$\left.\frac{\partial x}{\partial t}\right|_{(0,1)} = 2 \qquad \left.\frac{\partial y}{\partial t}\right|_{(0,1)} = s = 0$$

$$\frac{\partial V}{\partial s} = \cancelto{0}{\frac{\partial V}{\partial x}}\frac{\partial x}{\partial s} + \cancelto{5}{\frac{\partial V}{\partial y}}\cancelto{1}{\frac{\partial y}{\partial s}} = 5 \qquad \frac{\partial V}{\partial t} = \cancelto{0}{\frac{\partial V}{\partial x}}\frac{\partial x}{\partial t} + \cancelto{5}{\frac{\partial V}{\partial y}}\cancelto{0}{\frac{\partial y}{\partial t}} = 0$$

17) $f(x,y,z) = z e^{xy}$; $p = (0,1,2)$

$\nabla f = \langle yze^{xy}, xze^{xy}, e^{xy} \rangle$

$\nabla f = e^{xy}\langle yz, xz, 1 \rangle$

$\nabla f_p = \langle 2, 0, 1 \rangle \leftarrow$ direction.

$\|\nabla f_p\| = \sqrt{5} \leftarrow$ rate.

18) $z^2 = x^2 + y^2 \Rightarrow g(x,y,z) = z^2 - x^2 - y^2 = 0 \leftarrow$ Constraint

distace: $d = \sqrt{(x-2)^2 + (y-2)^2 \; z^2} \Rightarrow f(x,y,z) = (x-2)^2 + (y-2)^2 + z^2$, objective function.

$$\nabla f = \lambda \nabla g$$

$$\langle 2(x-2), 2(y-2), 2z \rangle = \lambda \langle -2x, -2y, 2z \rangle$$

$$\langle x-2, y-2, z \rangle = \lambda \langle -x, -y, z \rangle$$

$x - 2 = -\lambda x$ $\qquad$ $y - 2 = -\lambda y$ $\qquad$ $z = \lambda z$

$x(1+\lambda) = 2$ $\qquad$ $y(1+\lambda) = 2$ $\qquad$ $z(1-\lambda) = 0$

$z = 0$ or $\lambda = 1$

If $z = 0$ then $x = y = 0 \Rightarrow (0,0,0)$

If $\lambda = 1$ then $x = y = 1$ and $z = \pm\sqrt{2} \Rightarrow (1,1,-\sqrt{2})$ and $(1,1,\sqrt{2})$

$f(0,0,0) = 8$ $\qquad$ $f(1,1,\pm\sqrt{2}) = 4 \leftarrow$ minimum, so $(1,1,\pm\sqrt{2})$ are the closest points.

19) $f(x,y) = x^3 + y^2 - 12x + 6y - 7$

$f_x = 3x^2 - 12 \qquad f_y = 2y + 6 \qquad f_{xy} = f_{yx} = 0$

$f_{xx} = 6x \qquad f_{yy} = 2$

f_x and f_y are continuous on $\mathbb{R}^2$ so,

$$3x^2 - 12 = 0 \qquad 2y + 6 = 0$$
$$x^2 = 4 \qquad 2y = -6$$
$$x = \pm 2 \qquad y = -3 \quad \Rightarrow (-2,-3) \text{ and } (2,-3)$$

$\underline{(-2,-3):}$

$f_{xx} = -12 < 0$

$f_{yy} = 2$

$D = -12(2) = -24 < 0$

Saddle

$\underline{(2,-3):}$

$f_{xx} = 12 > 0$

$f_{yy} = 2$

$D = 12(2) = 24 > 0$

Minimum

20) $f(x,y,z) = \sqrt{x^2 - yz}$

a) $P = (3,2,4)$

$$\nabla f = \frac{\langle 2x, -z, -y \rangle}{2\sqrt{x^2 - yz}} \Rightarrow \nabla f_p = \frac{\langle 6, -4, -2 \rangle}{2\sqrt{9-8}} = \langle 3, -2, -1 \rangle$$

$$\vec{u} = \frac{\langle 3, -2, -1 \rangle}{\sqrt{14}}$$

b) $\|\nabla f_p\| = \sqrt{14}$

c) $3(x-3) - 2(y-2) - (z-4) = 0$

d) $\left.\frac{\partial z}{\partial y}\right|_p = -\left.\frac{f_y}{f_z}\right|_p = -\frac{-2}{-1} = -2$

e) $\sqrt{(3.1)^2-(1.9)(4.2)} \approx f(3,2,4) + 3(3.1-3) - 2(1.9-2) - (4.2-4)$

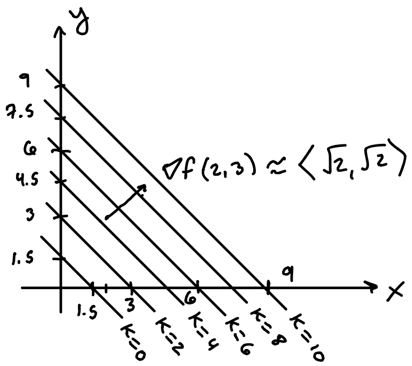
$\approx \sqrt{9-8} + 0.3 + 0.2 - 0.2$

$\approx 1 + 0.3$

≈ 1.3

21)

a)



$\|\nabla f(2,3)\| \approx 2$ since if moving from one contour to another increases f by $+2$ and the distace between adjacent contours is 1, thus $\|\nabla f(2,3)\| \approx \frac{2}{1} = 2$.

b) Let $\vec{u} = \langle 3,4 \rangle$.

$$D_{\vec{u}} f = \nabla f \cdot \frac{\vec{u}}{\|\vec{u}\|} = \frac{\langle \sqrt{2}, \sqrt{2} \rangle \cdot \langle 3,4 \rangle}{5} = \frac{7\sqrt{2}}{5}$$

c) $\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$

$= \nabla f(2,3) \cdot \langle 3,4 \rangle$

$= \langle \sqrt{2}, \sqrt{2} \rangle \cdot \langle 3,4 \rangle$

$= 7\sqrt{2}$

22) $f(x,y) = x^2 + 4xy + y^2 - 2x + 8y + 3$

$f_x = 2x + 4y - 2$ $\quad f_y = 4x + 2y + 8$ $\quad f_{xy} = f_{yx} = 4$

$f_{xx} = 2$ $\quad f_{yy} = 2$

f_x and f_y are continuous on $\mathbb{R}^2$ and so:

$2x + 4y - 2 = 0 \qquad 4x + 2y + 8 = 0$

$\Rightarrow -4x - 8y + 4 = 0 \Rightarrow +(-4x - 8y + 4 = 0)$

$-6y + 12 = 0$

$y = 2 \Rightarrow x = \frac{2-8}{2} = -3$

$\Rightarrow (-3,2)$ is the only critical point.

$\underline{(-3,2):}$

$f_{xx} = 2$

$f_{yy} = 2$

$f_{xy} = 4$

$D = 4 - 16 < 0 \Rightarrow$ saddle

23) $f(x,y) = 2x^2 + y^2 - 2x \leftarrow$ objective function

$g(x,y) = x^2 + y^2 - 4 = 0 \leftarrow$ constraint

Domain: $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\}$.

$f_x = 4x - 2 \qquad f_y = 2y \qquad f_{xy} = f_{yx} = 0$

$f_{xx} = 4 \qquad f_{yy} = 2 \qquad D = f_{xx} f_{yy} - f_{xy} = f_{xx} f_{yy}$

f_x and f_y are continuous on $\mathbb{R}^2$ so,

$4x - 2 = 0 \qquad 2y = 0$

$x = \frac{1}{2} \qquad y = 0 \quad \Rightarrow \left(\frac{1}{2}, 0\right)$ is the only critical point.

On the boundary:

$\nabla f = \lambda \nabla g$

$\langle 4x-2, 2y \rangle = \lambda \langle 2x, 2y \rangle$

$4x - 2 = 2\lambda x \qquad 2y = 2\lambda y$

$2y(1-\lambda) = 0$

$y = 0$ or $\lambda = 1$

① If $y = 0$ then $x = \pm 2$ and so:

$(-2, 0)$ and $(2, 0)$.

② If $\lambda = 1$ then

$4x - 2 = 2x$

$2x = 2$

$x = 1$ and $y = \pm\sqrt{3}$ and so,

$(1, -\sqrt{3})$ and $(1, \sqrt{3})$.

Compare values:

$f\left(\frac{1}{2}, 0\right) = \frac{1}{2} - 1 = -\frac{1}{2} \leftarrow$ Absolute Minimum

$f(\pm 2, 0) = 8 \mp 4 \Rightarrow f(2,0) = 4$ and $f(-2,0) = 12 \leftarrow$ Absolute maximum

$f(1, \pm\sqrt{3}) = 2 + 3 - 2 = 3$

24) $f(x,y) = 4-(x-1)(y-1)$ with $D = \{(x,y): 0 \le y \le 4-x^2\}$.

a) $f_x = 1-y$ $\quad f_y = 1-x \quad f_{xy} = f_{yx} = -1$

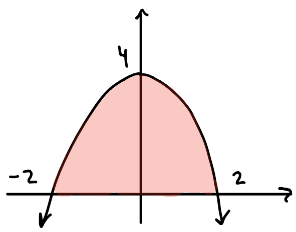
$f_{xx} = 0 \quad f_{yy} = 0 \quad D = -1 \leftarrow$ all critical points must be saddles.

f_x and f_y are continuous so:

$1-y=0 \quad 1-x=0$

$y=1 \quad x=1 \Rightarrow (1,1)$ is a saddle.

b)



D is both closed and bounded.

Restrict f to boundary component $y=0$:

$f(x,0) = 4+x-1 = x+3$

Potential extrema at $(-2,0)$ and $(2,0)$.

$f(-2,0) = 1$ and $f(2,0) = 5.$

c) $g(x,y) = y+x^2-4=0 \leftarrow$ constraint.

$f(x,y) = 4-(x-1)(y-1) \leftarrow$ objective function

$\nabla f = \lambda \nabla g$

$\langle 1-y, 1-x \rangle = \lambda \langle 2x, 1 \rangle$

$1-y = 2\lambda x \quad 1-x = \lambda$

$1-y = 2(1-x)x$

$1-y = 2x-2x^2$

$y = 1-2x+2x^2 \Rightarrow 1-2x+2x^2+x^2-4=0$

$3x^2-2x-3=0$

$x = \dfrac{2 \pm \sqrt{4-4(3)(-3)}}{6}$

$= \frac{1}{3} \pm \frac{1}{6}\sqrt{4+36}$

$x = \frac{1}{3} \pm \frac{1}{3}\sqrt{10}$

$$y = 1 - 2\left(\frac{1 \pm \sqrt{10}}{3}\right) + 2\left(\frac{1 \pm \sqrt{10}}{3}\right)^2$$

$$= 1 - \frac{2 \pm 2\sqrt{10}}{3} + 2\left(\frac{1 \pm 2\sqrt{10} + 10}{9}\right)$$

$$= \frac{9 - 6 \mp 6\sqrt{10} + 2 \pm 4\sqrt{10} + 20}{9}$$

$$y = \frac{25 \mp 2\sqrt{10}}{9}$$

And So,

$$P = \left(\frac{3 + 3\sqrt{10}}{9}, \frac{25 - 2\sqrt{10}}{9}\right) \quad \text{and} \quad Q = \left(\frac{3 - 3\sqrt{10}}{9}, \frac{25 + 2\sqrt{10}}{9}\right)$$

$$f(x,y) = 4 - (x-1)(y-1)$$

$$f\left(\frac{3 \pm 3\sqrt{10}}{9}, \frac{25 \mp 2\sqrt{10}}{9}\right) = 4 - \left(\frac{-6 \pm 3\sqrt{10}}{9}\right)\left(\frac{16 \mp 2\sqrt{10}}{9}\right)$$

$$= 4 - \frac{-96 \pm 12\sqrt{10} \pm 48\sqrt{10} - 60}{81}$$

$$= \frac{324 + 96 \mp 60\sqrt{10} + 60}{81}$$

$$= \frac{480 \mp 60\sqrt{10}}{81}$$

Note: you will not recieve an optimization problem that would require a Calculator or using something like approximation.

linear approximation: $\sqrt{10} \approx \sqrt{9} + \frac{1}{2\sqrt{9}}(1) = 3 + \frac{1}{6}$.

So,

$$f(P) = \frac{480 - 60\sqrt{10}}{81} \approx \frac{480}{81} - \frac{60}{81}\left(3 + \frac{1}{6}\right) = \frac{480 - 190}{81} = \frac{290}{81} = 3 + \frac{47}{81}$$

$$f(Q) = \frac{480 + 60\sqrt{10}}{81} \approx \frac{480}{81} + \frac{60}{81}\left(3 + \frac{1}{6}\right) = \frac{480 + 190}{81} = \frac{670}{81} = 8 + \frac{22}{81}$$

d) $f(P) \approx 3 + \frac{47}{81}$ $\quad f(1,1) = 4$ $\quad f(2,0) = 5$

$f(Q) \approx 8 + \frac{22}{81}$ ← maximum $\quad f(-2,0) = 1$ ← minimum.

25) $f(x,y) = x^3 - 6xy + 8y^3$

$f_x = 3x^2 - 6y$ $\quad f_y = -6x + 24y^2$ $\quad f_{xy} = -6 = f_{yx}$

$f_{xx} = 6x$ $\quad f_{yy} = 48y$

f_x and f_y are continuous so,

$3x^2 - 6y = 0$ $\quad -6x + 24y^2 = 0$

$x^2 - 2y = 0$ $\quad -6x + 24\left(\frac{x^4}{4}\right) = 0$

$y = \frac{1}{2}x^2$ $\quad -6x + 6x^4 = 0$

$6x(x^3 - 1) = 0$

$x = 0,\ x = 1 \Rightarrow (0,0)$ and $(1, \frac{1}{2})$

$\underline{(0,0)}$:

$f_{xx} = 0$

$f_{xy} = -6$

$D = -36 < 0 \Rightarrow$ Saddle

$f(0,0) = 0$ Saddle

$\underline{(1, \frac{1}{2})}$:

$f_{xx} = 6 > 0$ $\quad D = 6(24) - 36 > 0$

$f_{yy} = 24$ $\quad$ Minimum

$f_{xy} = -6$

$f(1, \frac{1}{2}) = -1$ local minimum.

26) $f(x,y) = e^{-x^2-y^2}(x^2 + 2y^2)$ $\quad D := \{(x,y) : x^2 + y^2 \leq 4\}.$

$f_x = -2xe^{-x^2-y^2}(x^2 + 2y^2) + 2xe^{-x^2-y^2}$

$f_x = (-2x^3 - 4xy^2 + 2x)e^{-x^2-y^2}$

$f_x = 2x(1 - x^2 - 2y^2)e^{-x^2-y^2}$

$f_y = -2ye^{-x^2-y^2}(x^2 + 2y^2) + e^{-x^2-y^2}(4y)$

$f_y = 2y(2 - x^2 - 2y^2)e^{-x^2-y^2}$

f_x and f_y are continuous on $\mathbb{R}^2$ so,

$$2x(1-x^2-2y^2)e^{-x^2-y^2}=0 \qquad 2y(2-x^2-2y^2)e^{-x^2-y^2}$$

$$x=0 \text{ or } x^2+2y^2=1 \qquad y=0 \text{ or } x^2+2y^2=2$$

① If $x=0$ then $y=0$ or $y=\pm 1$ $\Rightarrow$ $(0,0)$ and $(0,\pm 1)$

② If $x^2+2y^2=1$ then $y=0$ and $x=\pm 1$ $\Rightarrow$ $(\pm 1, 0)$

So, there are five critical points: $(0,0)$, $(0,\pm 1)$, and $(\pm 1, 0)$.

On the boundary of D: $x^2+y^2=4$ and so f restricted to the boundary becomes: $f(x,y)=e^{-4}(4+y^2)$ and so possible extrema: $(\pm 2, 0)$ and $(0,\pm 2)$.

$$f(0,0)=0 \qquad f(\pm 1,0)=e^{-1} \qquad f(0,\pm 2)=8e^{-4}$$

$$f(0,\pm 1)=2e^{-1} \qquad f(\pm 2,0)=4e^{-4}$$

Absolute minimum of 0 at $(0,0)$ and

Absolute maximum of $2e^{-1}$ at $(0,\pm 1)$

27) $f(x,y)=\frac{1}{x}+\frac{1}{y}$ ← objective function.

$g(x,y)=\frac{1}{x^2}+\frac{1}{y^2}-1=0$ ← constraint.

$$\nabla f = \lambda \nabla g$$

$$\left\langle -\frac{1}{x^2}, -\frac{1}{y^2}\right\rangle = \lambda \left\langle -\frac{2}{x^3}, -\frac{2}{y^3}\right\rangle$$

$$-\frac{1}{x^2}=-\frac{2\lambda}{x^3} \qquad -\frac{1}{y^2}=-\frac{2\lambda}{y^3}$$

$$x=2\lambda \qquad y=2\lambda \Rightarrow x=y \Rightarrow \frac{1}{y^2}+\frac{1}{y^2}=1$$

$$2=y^2$$

$$y=\pm\sqrt{2}$$

$$x=\pm\sqrt{2}$$

And so, $(\pm\sqrt{2}, \pm\sqrt{2})$

$$f(\pm\sqrt{2},\pm\sqrt{2}) = \pm\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}} = \pm\frac{2}{\sqrt{2}}$$

Absolute maximum: $f(\sqrt{2},\sqrt{2})=\frac{2}{\sqrt{2}}$

Absolute minimum: $f(-\sqrt{2},-\sqrt{2})=-\frac{2}{\sqrt{2}}$

28) Objective: $f(x,y,z) = x^2 + y^2 + z^2$

Constraint: $g(x,y,z) = xy^2z^3 - 2 = 0$

$$\nabla f = \lambda \nabla g$$

$$\langle 2x, 2y, 2z \rangle = \lambda \langle y^2z^3,\ 2xyz^3,\ 3xy^2z^2 \rangle$$

Since $xy^2z^3 = 2$, note that $x \neq 0$, $y \neq 0$, and $z \neq 0$.

$2x = \lambda y^2 z^3 \qquad 2y = 2\lambda xyz^3 \qquad 2z = 3\lambda xy^2z^2$

$\lambda = \dfrac{2x}{y^2z^3} \qquad \lambda = \dfrac{1}{xz^3} \qquad \lambda = \dfrac{2}{3xy^2z}$

$$\frac{2x}{y^2z^3} = \frac{1}{xz^3} \qquad \frac{1}{xz^3} = \frac{2}{3xy^2z}$$

$2x^2 = y^2$ and $3y^2 = 2z^2$ and $xy^2z^3 = 2$

$x^2 = \frac{1}{2}y^2 \qquad z^2 = \frac{3}{2}y^2 \qquad x^2y^4z^6 = 4$

$$\Rightarrow \frac{1}{2}y^2 y^4 \left(\frac{3}{2}\right)^3 y^6 = 4$$

$$y^{12} = \frac{2^2 \cdot 2^3 \cdot 2^1}{3^3}$$

$$y^{12} = \frac{2^6}{3^3}$$

$$y = \pm \frac{\sqrt{2}}{\sqrt[4]{3}}$$

$$y = \pm \sqrt[4]{\frac{4}{3}}$$

Note: $2^{1/2} = \left(\left(2^{1/2}\right)^{1/2}\right)^2 = \left(2^2\right)^{1/4} = \sqrt[4]{4}$

$x = \pm \frac{1}{\sqrt{2}}y$ and $z = \pm\sqrt{\frac{3}{2}}\,y$

Note: Since $xy^2z^3 = 2 > 0$, x and z must have the same sign.

① If $y = \sqrt[4]{\frac{4}{3}}$ then

$$x = \pm \frac{1}{\sqrt{2}}\sqrt[4]{\frac{4}{3}}$$

$$= \pm \frac{1}{\sqrt[4]{4}}\sqrt[4]{\frac{4}{3}}$$

Note: $\sqrt{3} = \sqrt[4]{9}$

$x = \pm \dfrac{1}{\sqrt[4]{3}}$ and $z = \pm \sqrt[4]{\frac{9}{4}}\sqrt[4]{\frac{4}{3}}$

$$z = \pm\sqrt[4]{3} \implies \left(\pm\frac{1}{\sqrt[4]{3}},\ \sqrt[4]{\frac{4}{3}},\ \pm\sqrt[4]{3}\right) := P_{\pm}$$

② If $y = -\sqrt[4]{\frac{4}{3}}$ then $x = \mp \frac{1}{\sqrt[4]{3}}$ and $z = \mp \sqrt[4]{3}$

$$\Rightarrow \left(\mp \frac{1}{\sqrt[4]{3}}, -\sqrt[4]{\frac{4}{3}}, \mp \sqrt[4]{3}\right) := Q_{\pm}$$

$$\left.\begin{aligned} f(P_{\pm}) &= \frac{1}{\sqrt{3}} + \sqrt{\frac{4}{3}} + \sqrt{3} = \frac{1+2+3}{\sqrt{3}} = \frac{5}{\sqrt{3}} \\ f(Q_{\pm}) &= \frac{1}{\sqrt{3}} + \sqrt{\frac{4}{3}} + \sqrt{3} = \frac{1+2+3}{\sqrt{3}} = \frac{5}{\sqrt{3}} \end{aligned}\right\} \text{Equal.}$$

The above point is a minimum since the implicit surface $xy^2z^3 = 2$ is unbounded and therefore points can be found as far away from the origin as desired.

For example, the point $\left(\sqrt[3]{2}\,k, \sqrt[3]{2}\,k, \frac{1}{k}\right)$ satisfies $xy^2z^3 = 2$ but as $k \longrightarrow \infty$ then $f\left(\sqrt[3]{2}\,k, \sqrt[3]{2}\,k, \frac{1}{k}\right) = 2\sqrt[3]{4}\,k^2 + \frac{1}{k^2} \longrightarrow \infty$

Recall: Absolute extrema of f are guaranteed to exist only if f is continuous and defined on a closed and bounded domain.

Here, $xy^2z^3 = 2$ implicitly defines the domain which is not bounded, and therefore f isn't guaranteed to attain both an absolute maximum and minimum.

Therefore, there is only a minimum $\underline{\underline{f(P_{\pm}) = f(Q_{\pm}) = \frac{5}{\sqrt{3}}}}$

29)