M273 Exam 3 Overview

This overview is provided to you as a brief listing of relevant formulas and information you'll likely need on the exam. This list is not exhaustive. You may or may not need everything on this list to succeed on the exam.

The formulas in the box will be provided on the exam.

$dA = r\,dr\,d\theta$	$dV = r\,dz\,dr\,d\theta$	$dV = \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$

15.1 Double Integration Over Rectangles

- Fubini's Theorem: integrating a continuous function f over a rectangle $\mathcal{R} = [a, b] \times [c, d]$ can be determined by evaluating an iterated integral in either order

$$\iint_{\mathcal{R}} f(x, y)\,dA = \int_a^b \int_c^d f(x, y)\,dy\,dx = \int_c^d \int_a^b f(x, y)\,dx\,dy$$

15.2 Double Integration Over More General Regions

- Vertically simple region $\mathcal{D} : a \le x \le b$ and $g_1(x) \le y \le g_2(x)$

$$\iint_{\mathcal{D}} f(x, y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\,dy\,dx$$

- Horizontally simple region $\mathcal{D} : c \le y \le d$ and $g_1(y) \le x \le g_2(y)$

$$\iint_{\mathcal{D}} f(x, y)\,dA = \int_c^d \int_{g_1(y)}^{g_2(y)} f(x, y)\,dx\,dy$$

- Volume between two surfaces. Let $z = z_1(x, y)$ and $z = z_2(x, y)$ be two surfaces such that $z_1(x, y) \le z \le z_2(x, y)$ for all $z \in \mathcal{D}$ where $\mathcal{D}$ is the projection of the bounded region onto the xy-plane, then the volume bounded between these two surfaces is given by

$$\iint_{\mathcal{D}} (z_2(x, y) - z_1(x, y))\,dA$$

15.4 Part I Double Integration in Polar Coordinates

- Polar conversions: $x = r\cos\theta$ and $y = r\sin\theta$
- Polar differential area element: $dA = r\,dr\,d\theta$
- Radially simple region $\mathcal{D} : \theta_1 \le \theta \le \theta_2$ and $r_1(\theta) \le r \le r_2(\theta)$

$$\iint_{\mathcal{D}} f(x, y)\,dA = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta) r\,dr\,d\theta$$

15.3 Triple Integrals in Cartesian Coordinates

- Fubini's theorem: integrating a continuous function f over a box $\mathcal{B} = [a,b] \times [c,d] \times [p,q]$ can be determined by evaluating an iterated integral in any order (there are $3! = 6$ possible orders)

- Integrating over a z-simple region $\mathcal{W} : (x,y) \in \mathcal{D}$ where $z_1(x,y) \leq z \leq z_2(x,y)$ where $\mathcal{D}$ is the projection of $\mathcal{W}$ onto the xy-plane:

$$\iiint\limits_{\mathcal{W}} f(x,y,z)\,dV = \iint\limits_{\mathcal{D}} \left(\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz \right) dA$$

- Integrating over a y-simple region $\mathcal{W} : (x,z) \in \mathcal{D}$ where $y_1(x,y) \leq y \leq y_2(x,z)$ where $\mathcal{D}$ is the projection of $\mathcal{W}$ onto the xz-plane:

$$\iiint\limits_{\mathcal{W}} f(x,y,z)\,dV = \iint\limits_{\mathcal{D}} \left(\int_{y_1(x,z)}^{y_2(x,z)} f(x,y,z)\,dy \right) dA$$

- Integrating over a x-simple region $\mathcal{W} : (y,z) \in \mathcal{D}$ where $x_1(y,z) \leq x \leq x_2(y,z)$ where $\mathcal{D}$ is the projection of $\mathcal{W}$ onto the yz-plane:

$$\iiint\limits_{\mathcal{W}} f(x,y,z)\,dV = \iint\limits_{\mathcal{D}} \left(\int_{x_1(y,z)}^{x_2(y,z)} f(x,y,z)\,dx \right) dA$$

15.4 Part II Triple Integration in Cylindrical and Spherical Coordinates

- Cylindrical conversions: $x = r\cos\theta$, $y = r\sin\theta$, and z remains the same
- Cylindrical differential volume element: $dV = r\,dz\,dr\,d\theta$
- Cylindrically simple region $\mathcal{W} : \theta_1 \leq \theta \leq \theta_2$, $r_1(\theta) \leq r \leq r_2(\theta)$, and $z_1(r,\theta) \leq z \leq z_2(r,\theta)$

$$\iiint\limits_{\mathcal{W}} f(x,y,z)\,dV = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} \int_{z_1(r,\theta)}^{z_2(r,\theta)} f(r\cos\theta, r\sin\theta, z) r\,dz\,dr\,d\theta$$

- Spherical conversions: $x = \rho\cos\theta\sin\phi$, $y = \rho\sin\theta\sin\phi$, and $z = \rho\cos\phi$
- Spherical differential volume element: $dV = \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$
- Spherically simple region $\mathcal{W} : \theta_1 \leq \theta \leq \theta_2$, $\phi_1 \leq \phi \leq \phi_2$, and $\rho_1(\theta,\phi) \leq \rho \leq \rho_2(\theta,\phi)$

$$\iiint\limits_{\mathcal{W}} f(x,y,z)\,dV = \int_{\theta_1}^{\theta_2} \int_{\phi_1}^{\phi_2} \int_{\rho_1(\theta,\phi)}^{\rho_2(\theta,\phi)} f(\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)\rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$$

Useful Trigonometric Identities

- $\sin^2\theta + \cos^2\theta = 1$
- $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$
- $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$

16.1 Vector Fields

- "Nabla" $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$
- Divergence operator: $\text{div}(\mathbf{F}) = \nabla \cdot \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle F_1, F_2, F_3 \rangle = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$
- Curl operator: $\text{curl}(\mathbf{F}) = \nabla \times \mathbf{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle$
- A vector field $\mathbf{F}$ is conservative if there exists a scalar potential function f such that $\nabla f = \mathbf{F}$

16.2 Line Integrals

- A path $\mathcal{C}$ in $\mathbb{R}^3$ can be parameterized by $\mathbf{r}(t)$ for $t \in [a, b]$
- Scalar line integral of a scalar function f over a path $\mathcal{C}$

$$\int_{\mathcal{C}} f(x, y, z)\, ds = \int_a^b f(\mathbf{r}(t))\, \|\mathbf{r}'(\mathbf{t})\|\, dt$$

- If $f(x, y, z) = 1$ then the scalar line integral of f over a path $\mathcal{C}$ is the length of $\mathcal{C}$, i.e. the arc length,

$$\int_{\mathcal{C}} ds = \int_a^b \|\mathbf{r}'(\mathbf{t})\|\, dt = \text{length}(\mathcal{C})$$

- Vector line integral of a vector field $\mathbf{F}$ over a path $\mathcal{C}$

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}} (\mathbf{F} \cdot \mathbf{T})\, ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

- Alternate notation for a vector line integral of $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$ over a path $\mathcal{C}$

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}} F_1\, dx + F_2\, dy + F_3\, dz$$

16.3 Conservative Vector Fields

- The vector line integral over a closed path (endpoints are equal) $\mathcal{C}$ is called the circulation and is denoted

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

- If $\mathbf{F}$ is conservative, that is $\mathbf{F} = \nabla f$ for some scalar function f, and $\mathcal{C}$ is a path with endpoints P and Q then

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}} \nabla f \cdot d\mathbf{r} = f(Q) - f(P)$$

- If $\mathbf{F}$ is conservative, that is $\mathbf{F} = \nabla f$ for some scalar function f, and $\mathcal{C}$ is a closed path, then

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = 0$$

- If $\operatorname{curl}(\mathbf{F}) = \mathbf{0}$ (or if $\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y} = 0$ for 2D fields) and $\mathbf{F}$ is defined on a simply connected domain, then $\mathbf{F}$ is conservative and therefore, there exists a scalar potential function f such that $\nabla f = \mathbf{F}$

- If $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$ is known to be conservative then f can be found by evaluating the following antiderivatives:

$$f = \int F_1 \, dx \qquad f = \int F_2 \, dy \qquad f = \int F_3 \, dz$$

17.1 Green's Theorem

- Let $\mathcal{D}$ be domain whose boundary $\partial\mathcal{D}$ is a simple closed curve oriented counterclockwise, then

$$\oint_{\partial\mathcal{D}} \mathbf{F} \cdot d\mathbf{r} = \iint_{\mathcal{D}} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA$$

- When orienting a boundary, if $\mathcal{D}$ lies to the left as the boundary is traversed then this is considered to be oriented positively

- The area of $\mathcal{D}$ can be determined using a line integral around the boundary $\partial\mathcal{D}$ provided

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$$

  A few common fields that have this property are

$$\mathbf{F} = \langle 0, x \rangle \qquad \mathbf{F} = \langle -y, 0 \rangle \qquad \mathbf{F} = \left\langle -\frac{y}{2}, \frac{x}{2} \right\rangle$$

- If the boundary of $\mathcal{D}$ is composed of multiple curves, then the total boundary $\partial\mathcal{D}$ can be written as a sum or difference of the constituent curves. For example, if $\partial\mathcal{D}$ is composed of two boundaries $\mathcal{C}_1$ and $\mathcal{C}_2$ then $\partial\mathcal{D} = \pm\mathcal{C}_1 \pm \mathcal{C}_2$ where the choice of plus or minus depends on whether that curve has been oriented positively or negatively. Positively oriented curves get a plus sign and negatively oriented curves receive the minus sign.