Exam III Review Solutions:

1) $\nabla \cdot \vec{F}$ is a scalar.

$\nabla(\nabla \cdot \vec{F})$ is a vector.

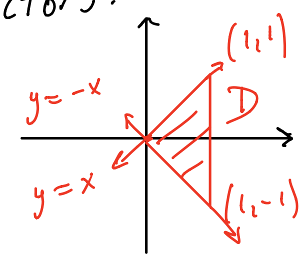
$\nabla \times (\nabla \times \vec{F})$ is a vector

$(\nabla f) \times \vec{F}$ is a vector

$(\nabla \cdot \vec{F}) \cdot \vec{F}$ is nonsense:

$\nabla \cdot \vec{F}$ is a scalar and the dot product is an operation defined between two vectors.

2) $\iint_D x \, dA$



D is vertically simple:

$0 \leq x \leq 1$

$-x \leq y \leq x$

$$\int_{x=0}^{1} \int_{y=-x}^{x} x \, dy \, dx = \int_0^1 xy \Big|_{-x}^{x} dx$$

$$= \int_0^1 \left[x^2 + x^2 \right] dx$$

$$= \frac{2}{3} x^3 \Big|_0^1$$

$$= \frac{2}{3}$$

3) $\displaystyle\int_{x=1}^{e}\int_{y=0}^{\frac{1}{x}} xy\,e^{xy}\,dy\,dx$

$u = xy \qquad v = \frac{1}{x}e^{xy}$

$du = x\,dy \qquad dv = e^{xy}\,dy$

$$= \int_1^e \left[\frac{\cancel{x}y}{\cancel{x}} e^{xy}\Big|_0^{\frac{1}{x}} - \int_0^{\frac{1}{x}} \frac{\cancel{x}}{\cancel{x}} e^{xy}\,dy \right] dx$$

$$= \int_1^e \left[y e^{xy}\Big|_0^{\frac{1}{x}} - \int_0^{\frac{1}{x}} e^{xy}\,dy \right] dx$$

$$= \int_1^e \left[\frac{1}{x}e^{\cancel{\frac{x}{x}}} - 0 - \frac{1}{x}e^{xy}\Big|_0^{\frac{1}{x}} \right] dx$$

$$= \int_1^e \left(\frac{e}{x} - \frac{1}{x}e^{\cancel{\frac{x}{x}}} + \frac{1}{x}e^0 \right) dx$$

$$= \int_1^e \left(\cancel{\frac{e}{x}} - \cancel{\frac{e}{x}} + \frac{1}{x} \right) dx$$

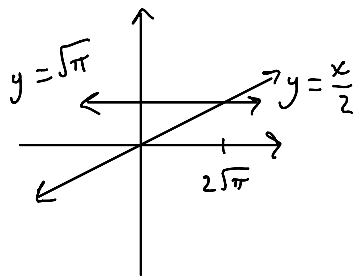
$$= \int_1^e \frac{1}{x}\,dx$$

$$= \ln|x|\Big|_1^e$$

$$= \cancelto{1}{\ln e} - \cancelto{0}{\ln 1}$$

$$= \underline{\underline{1}}$$

4) $\displaystyle\int_{x=0}^{2\sqrt{\pi}}\int_{y=x/2}^{\sqrt{\pi}} \sin(y^2)\,dy\,dx$ Can't integrate in the given order. Sketch and then reinterpret the region as horizontally simple.



V.S. $0 \le x \le 2\sqrt{\pi}$ H.S. $0 \le y \le \sqrt{\pi}$

$\frac{x}{2} \le y \le \sqrt{\pi}$ $0 \le x \le 2y$

$$\int_{x=0}^{2\sqrt{\pi}}\int_{y=x/2}^{\sqrt{\pi}} \sin(y^2)\,dy\,dx = \int_{y=0}^{\sqrt{\pi}}\int_{x=0}^{2y} \sin(y^2)\,dx\,dy$$

$$= \int_0^{\sqrt{\pi}} x\sin(y^2)\Big|_0^{2y}\,dy$$

$$= \int_0^{\sqrt{\pi}} 2y\sin(y^2)\,dy$$

$u = y^2$ $du = 2y\,dy$

$u(0) = 0$ $u(\sqrt{\pi}) = \pi$

$$= \int_0^{\pi} \sin u\,du$$

$$= -\cos u\Big|_0^{\pi}$$

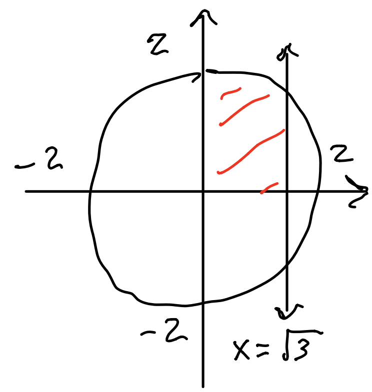
$$= -\cos\pi + \cos 0$$

$$= 1 + 1$$

$$= 2$$

5) $W = \{(x,y,z) : x^2+y^2+z^2 \leq 4,\ 0 \leq x \leq \sqrt{3},\ y,z \geq 0\}$.

$$\iiint_W xz\,dV \qquad \begin{aligned} 0 &\leq z \leq \sqrt{4-x^2-y^2} \\ 0 &\leq y \leq \sqrt{4-x^2} \\ 0 &\leq x \leq \sqrt{3} \end{aligned}$$



$$= \int_{x=0}^{\sqrt{3}} \int_{y=0}^{\sqrt{4-x^2}} \int_{z=0}^{\sqrt{4-x^2-y^2}} xz\,dz\,dy\,dx$$

$$= \int_{x=0}^{\sqrt{3}} \int_{y=0}^{\sqrt{4-x^2}} \frac{1}{2} x z^2 \Big|_0^{\sqrt{4-x^2-y^2}} dy\,dx$$

$$= \int_{x=0}^{\sqrt{3}} \int_{y=0}^{\sqrt{4-x^2}} \frac{1}{2} x \left(4-x^2-y^2\right) dy\,dx$$

$$= \frac{1}{2} \int_{x=0}^{\sqrt{3}} \int_{y=0}^{\sqrt{4-x^2}} \left(4x - x^3 - xy^2\right) dy\,dx$$

$$= \frac{1}{2} \int_0^{\sqrt{3}} \left[4xy - x^3 y - \frac{1}{3} x y^3 \Big|_0^{\sqrt{4-x^2}} \right] dx$$

$$= \frac{1}{2}\int_0^{\sqrt{3}} \left(4x\sqrt{4-x^2} - x^3\sqrt{4-x^2} - \frac{1}{3}x\left(4-x^2\right)^{3/2}\right) dx$$

$$= \frac{1}{2}\int_0^{\sqrt{3}} \left(4x\sqrt{4-x^2} - x^3\sqrt{4-x^2} - \frac{1}{3}x\left(4-x^2\right)^{3/2}\right) dx$$

$$= \frac{1}{2}\int_0^{\sqrt{3}} x\left(4\sqrt{4-x^2} - x^2\sqrt{4-x^2} - \frac{1}{3}\left(4-x^2\right)^{3/2}\right) dx$$

$$= \frac{1}{2}\int_0^{\sqrt{3}} x\left(\left(4-x^2\right)\sqrt{4-x^2} - \frac{1}{3}\left(4-x^2\right)^{3/2}\right) dx$$

$$= \frac{1}{\cancel{2}}\left(\frac{\cancel{2}}{3}\right)\int_0^{\sqrt{3}} x\left(4-x^2\right)^{3/2} dx$$

$$u = 4-x^2$$
$$du = -2x\,dx$$
$$-\frac{1}{2}du = x\,dx$$
$$u(0) = 4 \quad u(\sqrt{3}) = 1$$

$$-\frac{1}{2}\left(\frac{1}{3}\right)\int_4^1 u^{3/2} du$$

$$= \frac{1}{6}\int_1^4 u^{3/2} du$$

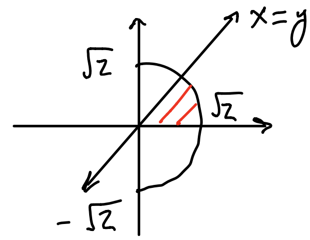
$$= \frac{1}{6}\left(\frac{2}{5}\right)\left[u^{5/2}\Big|_1^4\right]$$

$$= \frac{1}{15}\left[32 - 1\right] = \frac{31}{15}$$

6) $$\int_{y=0}^{1}\int_{x=y}^{\sqrt{2-y^2}} (x^2+y^2)\,dx\,dy$$

$0 \le y \le 1$

$y \le x \le \sqrt{2-y^2}$



As a radially simple region:

$0 \le \theta \le \frac{\pi}{4}$

$0 \le r \le \sqrt{2}$

$$= \int_{\theta=0}^{\frac{\pi}{4}}\int_{r=0}^{\sqrt{2}} \left(r^2\cos^2\theta + r^2\sin^2\theta\right) r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} d\theta \int_0^{\sqrt{2}} r^3\,dr$$

$$= \theta\Big|_0^{\frac{\pi}{4}}\ \frac{1}{4}r^4\Big|_0^{\sqrt{2}}$$

$$= \left(\frac{\pi}{4}-0\right)\frac{1}{\cancel{4}}\left(\cancel{4}-0\right)$$

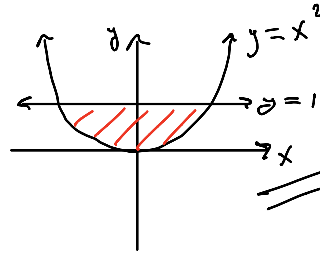
$$= \frac{\pi}{4}$$

7) $$\int_{x=-1}^{1}\int_{y=x^2}^{1} \left(1+y^{3/2}\right)^5 dy\,dx$$

$-1 \le x \le 1$

$x^2 \le y \le 1$

a)



b) $0 \le y \le 1$

$-\sqrt{y} \le x \le \sqrt{y}$

$$\int_{y=0}^{1}\int_{x=-\sqrt{y}}^{\sqrt{y}} \left(1+y^{3/2}\right)^5 dx\,dy$$

c) $$\int_0^1 x\left(1+y^{3/2}\right)^5\Big|_{-\sqrt{y}}^{\sqrt{y}}\,dy$$

$$= \int_0^1 2\sqrt{y}\left(1+y^{3/2}\right)^5 dy$$

$u = 1+y^{3/2}$

$du = \frac{3}{2}y^{1/2}\,dy$

$\frac{2}{3}du = y^{1/2}\,dy$

$u(0)=1$

$u(1)=2$

$$= 2\left(\frac{2}{3}\right)\int_1^2 u^5\,du$$

$$= \frac{4}{3}\left(\frac{1}{6}\right)u^6\Big|_1^2$$

$$= \frac{4}{18}\left[64-1\right]$$

$$= \frac{126}{9}$$

$$= 14$$

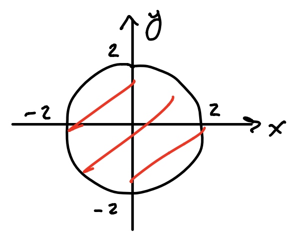
8) $z = x^2+y^2$ and $z = 8-x^2-y^2$ Note: $x^2+y^2 \le 8-x^2-y^2$

a) The curve for intersection:

$$x^2+y^2 = 8-x^2-y^2$$

$$2x^2+2y^2 = 8$$

$$x^2+y^2 = 4$$



$-2 \le x \le 2$

$-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$

$$V = \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=x^2+y^2}^{8-x^2-y^2} 1\,dz\,dy\,dx$$

b) Note: $x^2+y^2 = r^2$ and $8-x^2-y^2 = 8-r^2$

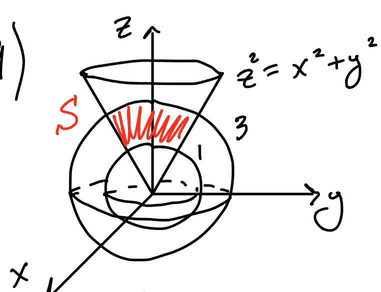
$0 \le \theta \le 2\pi$

$0 \le r \le 2$

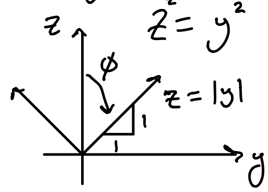
$r^2 \le z \le 8-r^2$

$$V = \int_{\theta=0}^{2\pi}\int_{r=0}^{2}\int_{z=r^2}^{8-r^2} r\,dz\,dr\,d\theta$$

9)



Projection of the cone into the yz-plane:



$\Rightarrow$ $z = |y|$ Slope of 1,

so $\phi_{max} = \frac{\pi}{4}$, i.e.

$0 \le \phi \le \frac{\pi}{4}$

Region is symmetric about the z-axis so, $0 \le \theta \le 2\pi$

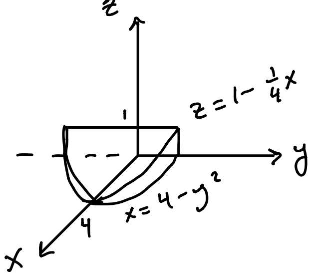
and $1 \le \rho \le 3$.

$$\mathrm{Vol}(S) = \iiint_S dV$$

$$= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\frac{\pi}{4}} \int_{\rho=1}^{3} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

10) $$\int_{x=0}^{4} \int_{y=-\sqrt{4-x}}^{\sqrt{4-x}} \int_{z=0}^{1-\frac{1}{4}x} f(x,y,z)\, dz\, dy\, dx$$

$$W: \begin{cases} 0 \le x \le 4 \\ -\sqrt{4-x} \le y \le \sqrt{4-x} \Rightarrow \begin{cases} y^2 = 4-x \\ x = 4-y^2 \end{cases} \\ 0 \le z \le 1-\frac{1}{4}x \Rightarrow \begin{cases} z = 0 \\ z = 1-\frac{1}{4}x \end{cases} \end{cases}$$

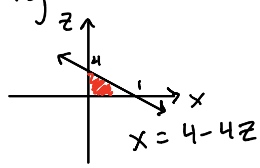


dy dx dz: y-simple:

$$x = 4-y^2 \Rightarrow y = \pm\sqrt{4-x}$$

$$-\sqrt{4-x} \le y \le \sqrt{4-x}$$

Projection of W into the xz-plane:

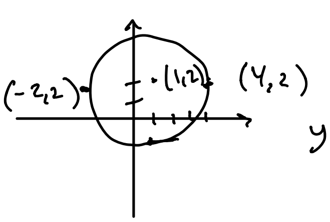


$$0 \le z \le 4$$
$$0 \le x \le 4-4z$$

$$\int_{z=0}^{4} \int_{x=0}^{4-4z} \int_{y=-\sqrt{4-x}}^{\sqrt{4-x}} f(x,y,z)\, dy\, dx\, dz$$

11) $x^2+y^2 \le z \le 2x+4y+4$. The two surfaces intersect when:

$$x^2+y^2 = 2x+4y+4$$
$$x^2-2x+1+y^2-4y+4 = 4+1+4$$
$$(x-1)^2+(y-2)^2 = 9$$ ← Circle of radius 3 centered at (1,2).



$$(y-2)^2 = 9-(x-1)^2$$
$$y-2 = \pm\sqrt{9-(x-1)^2}$$
$$y = 2 \pm \sqrt{9-(x-1)^2}$$

$$V = \int_{x=-2}^{4} \int_{y=2-\sqrt{9-(x-1)^2}}^{2+\sqrt{9-(x-1)^2}} \int_{x^2+y^2=z}^{2x+4y+4} dz\, dy\, dx$$

12) ∇f is a vector.

$\nabla \vec{F}$ is nonsense. ∇ acts on a Scalar to produce a vector.

$\nabla \times \vec{F}$ is a vector.

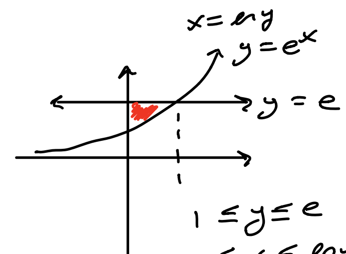
$\nabla(\nabla \times \vec{F})$ is nonsense. ∇ acts on a Scalar to produce a vector, but $\nabla \times \vec{F}$ is a vector.

$\nabla \cdot f$ is nonsense. the divergence operator acts on vector fields.

13) $$\int_{x=0}^{1}\int_{y=e^x}^{e} f(x,y)\,dy\,dx = \int_{y=1}^{e}\int_{x=0}^{\ln y} f(x,y)\,dx\,dy$$

$0 \le x \le 1$

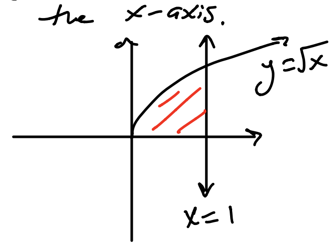
$e^x \le y \le e$



$1 \le y \le e$

$0 \le x \le \ln y$

14) $\iint_D y(1+x^2)^{-1}\,dA$ where D is the region bounded by $y=\sqrt{x}$, $x=1$, and the x-axis.



Vertically Simple: $0 \le x \le 1$, $0 \le y \le \sqrt{x}$

Horizontally Simple: $0 \le y \le 1$, $y^2 \le x \le 1$

$$\int_{x=0}^{1}\int_{y=0}^{\sqrt{x}} y(1+x^2)^{-1}\,dy\,dx$$

$$= \frac{1}{2}\int_0^1 y^2(1+x^2)^{-1}\Big|_0^{\sqrt{x}}\,dx$$

$$= \frac{1}{2}\int_0^1 x(1+x^2)^{-1}\,dx$$

$u = 1+x^2 \qquad u(0) = 1$

$du = 2x\,dx \qquad u(1) = 2$

$\frac{1}{2}du = x\,dx$

$$= \frac{1}{2}\left(\frac{1}{2}\right)\int_1^2 u^{-1}\,du$$

$$= \frac{1}{4}\ln|u|\Big|_1^2$$

$$= \frac{1}{4}\ln 2 - \cancel{\frac{1}{4}\ln 1}^{\,0}$$

$$= \frac{1}{4}\ln 2$$

15) $\iint_D e^{y^2}\,dA$ where D is the region bounded by: $\frac{x}{2} \le y \le 2$, $0 \le x \le 4$.

$$= \int_{x=0}^{4}\int_{y=\frac{x}{2}}^{2} e^{y^2}\,dy\,dx$$

but e^{y^2} doesn't have an elementary antiderivative. Try switching the order of integration. Sketch D:

$$\iint_D e^{y^2}\,dA = \int_{y=0}^{2}\int_{x=0}^{2y} e^{y^2}\,dx\,dy$$

$$= \int_0^2 x e^{y^2}\Big|_0^{2y} dy$$

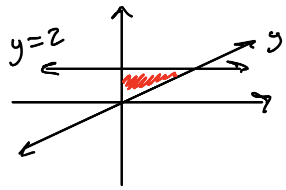
$$= \int_0^2 2y\, e^{y^2}\,dy$$

$$= \int_0^4 e^u\,du$$

$$= e^u\Big|_0^4$$

$$= e^4 - e^0$$

$$= e^4 - 1$$



$0 \le y \le 2$

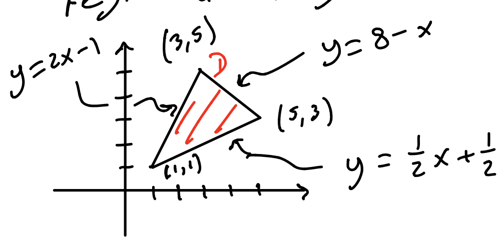
$0 \le x \le 2y$

$u = y^2 \qquad u(0) = 0$

$du = 2y\,dy \qquad u(2) = 4$

16) $z = x+1$ Triangle: $(1,1)$, $(5,3)$, and $(3,5)$

Where does $z = x+1$ intersect the xy-plane? $0 = x+1 \Rightarrow x = -1$. (thankfully) $x=-1$ doesn't intersect / go through the triangular region of integration.



$$y - 1 = \frac{5-1}{3-1}(x-1)$$
$$y = 2(x-1)+1$$
$$\boxed{y = 2x-1}$$

$$y - 1 = \frac{3-1}{5-1}(x-1)$$
$$y = \tfrac{1}{2}(x-1)+1$$
$$\boxed{y = \tfrac{1}{2}x + \tfrac{1}{2}}$$

$$y - 5 = \frac{3-5}{5-3}(x-3)$$
$$y = -\tfrac{2}{2}(x-3)+5$$
$$\boxed{y = 8-x}$$

__Notice:__ D is neither vertically nor horizontally simple, so multiple integrals will be required regardless.

$1 \le x \le 3 \qquad\qquad 3 \le x \le 5$

$\tfrac{1}{2}x+\tfrac{1}{2} \le y \le 2x-1 \qquad \tfrac{1}{2}x+\tfrac{1}{2} \le y \le 8-x$

$$V = \int_{x=1}^{3}\int_{y=\frac{1}{2}x+\frac{1}{2}}^{2x-1}\int_{z=0}^{x+1} dz\,dy\,dx + \int_{x=3}^{5}\int_{y=\frac{1}{2}x+\frac{1}{2}}^{8-x}\int_{z=0}^{x+1} dz\,dy\,dx$$

$$= \int_{x=1}^{3}\int_{y=\frac{1}{2}x+\frac{1}{2}}^{2x-1}(x+1)\,dy\,dx + \int_{x=3}^{5}\int_{y=\frac{1}{2}x+\frac{1}{2}}^{8-x}(x+1)\,dy\,dx$$

$$= \int_1^3 \left[xy + y \Big|_{\frac{1}{2}x+\frac{1}{2}}^{2x-1} \right] dx + \int_3^5 \left[xy + y \Big|_{\frac{1}{2}x+\frac{1}{2}}^{8-x} \right] dx$$

$$= \int_1^3 \left[x(2x-1) + 2x - 1 - x\left(\tfrac{1}{2}x + \tfrac{1}{2}\right) - \tfrac{1}{2}x - \tfrac{1}{2} \right] dx$$

$$+ \int_3^5 \left[x(8-x) + 8 - x - x\left(\tfrac{1}{2}x + \tfrac{1}{2}\right) - \tfrac{1}{2}x - \tfrac{1}{2} \right] dx$$

$$= \int_1^3 \left[2x^2 - x + 2x - 1 - \tfrac{1}{2}x^2 - \tfrac{1}{2}x - \tfrac{1}{2}x - \tfrac{1}{2} \right] dx$$

$$+ \int_3^5 \left[8x - x^2 + 8 - x - \tfrac{1}{2}x^2 - \tfrac{1}{2}x - \tfrac{1}{2}x - \tfrac{1}{2} \right] dx$$

$$= \int_1^3 \left(\tfrac{3}{2}x^2 - \tfrac{3}{2} \right) dx + \int_3^5 \left(-\tfrac{3}{2}x^2 + 6x + \tfrac{15}{2} \right) dx$$

$$= \left[\tfrac{1}{2}x^3 - \tfrac{3}{2}x \Big|_1^3 \right] + \left[-\tfrac{1}{2}x^3 + 3x^2 + \tfrac{15}{2}x \Big|_3^5 \right]$$

$$= \frac{27}{2} - \frac{9}{2} - \frac{1}{2} + \frac{3}{2} - \frac{125}{2} + 75 + \frac{75}{2}$$

$$+ \frac{27}{2} - 27 - \frac{45}{2}$$

$$= \frac{27 - 9 - 1 + 3 - 125 + 75 + 27 - 45}{2} + 75 - 27$$

$$= -24 + 48$$

$$= 24$$

17) a) $\int_C (x+y)\, ds$ where C is the straight line segment from $(1,2)$ to $(4,6)$.

$C: \vec{r}(t) = (1-t)\langle 1,2\rangle + t\langle 4,6\rangle$

$= \langle 1-t, 2-2t\rangle + \langle 4t, 6t\rangle$

$= \langle 1+3t, 2+4t\rangle$ for $t \in [0,1]$.

$\vec{r}\,'(t) = \langle 3,4\rangle \Rightarrow \|\vec{r}\,'(t)\| = 5$

$$\int_C (x+y)\,ds = \int_0^1 (1+3t+2+4t)\,5\,dt$$

$$= 5\int_0^1 (7t+3)$$

$$= 5\left[\frac{7}{2}t^2 + 3t \Big|_0^1\right]$$

$$= 5\left(\frac{7}{2}+3\right)$$

$$= \frac{35}{2} + 15$$

$$= \frac{35+30}{2}$$

$$= \frac{65}{2}$$

b) $\int_C x\,dy - y\,dx$ where C is the semicircle $x^2+y^2=4$ for $y \geq 0$, oriented counterclockwise.

$\vec{r}(t) = \langle 2\cos t, 2\sin t\rangle$ for $t \in [0,\pi]$ $\quad \vec{r}\,'(t) = \langle -2\sin t, 2\cos t\rangle$

$$= \int_0^{\pi} (2\cos t \cdot 2\cos t + 2\sin t \cdot 2\sin t)\,dt$$

$$= \int_0^{\pi} (4\cos^2 t + 4\sin^2 t)\,dt$$

$$= 4\int_0^{\pi} dt$$

$$= \underline{\underline{4\pi}}$$

c) $\displaystyle\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = \langle 2x+y, x-2y\rangle$

<u>Note</u>: $\vec{F}$ is conservative: $\dfrac{\partial F_2}{\partial x} = 1 = \dfrac{\partial F_1}{\partial y}$

which means $\displaystyle\int_C \vec{F}\cdot d\vec{r}$ is path independent and:

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(2,1) - f(0,0),$$ where f is a potential function for $\vec{F}$.

$$\nabla f = \langle 2x+y, x-2y\rangle$$

$$\Rightarrow \quad \frac{\partial f}{\partial x} = 2x+y \qquad\qquad \frac{\partial f}{\partial y} = x-2y$$

$$\int \partial f = \int (2x+y)\,\partial x \qquad\qquad \int \partial f = \int (x-2y)\,\partial y$$

$$f = x^2 + xy + g(y) \qquad\qquad f = xy - y^2 + h(x)$$

$$\Rightarrow \quad f(x,y) = x^2 + xy - y^2$$

<u>Check</u>: $\nabla f = \langle 2x+y, x-2y\rangle$ ✓

Thus,

$$\int_C \vec{F}\cdot d\vec{r} = f(2,1) - f(0,0)$$

$$= 4 + 2 - 1 - 0$$

$$= \underline{\underline{5}}$$

18) $\int_C ds$ where the curve the C is given by:

$\vec{r}(t) = \langle 4t, -3t, 12t \rangle$ for $t \in [2,5]$.

$\vec{r}\,'(t) = \langle 4, -3, 12 \rangle \Rightarrow \|\vec{r}\,'(t)\| = \sqrt{16+9+144}$

$= \sqrt{169}$

$= 13$

$= \int_2^5 13\,dt$

$= 13(5-2)$

$= 39$ This is the arc length of C

19) $\vec{r}(t) = \langle \cos t, \sin t, t^2 \rangle$ for $t \in [0, 2\pi]$ given mass density:

$\rho(x,y,z) = \sqrt{z}$ g/cm.

$\vec{r}\,'(t) = \langle -\sin t, \cos t, 2t \rangle$

$\|\vec{r}\,'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 4t^2}$

$= \sqrt{1+4t^2}$

$M = \int_C \rho(x,y,z)\,ds$

$= \int_0^{2\pi} \sqrt{t^2}\,\sqrt{1+4t^2}\,dt$

$= \int_0^{2\pi} t\sqrt{1+4t^2}\,dt$

$u = 1+4t^2$

$du = 8t\,dt$

$\frac{1}{8}du = t\,dt$

$u(0) = 1$

$u(2\pi) = 1 + 16\pi^2$

$= \frac{1}{8}\int_1^{1+16\pi^2} u^{1/2}\,du$

$= \frac{1}{8}\left(\frac{2}{3}\right)\left[u^{3/2} \Big|_1^{1+16\pi^2} \right]$

$= \frac{1}{12}\left[(1+16\pi^2)^{3/2} - 1 \right]$

20) $\vec{F} = \langle x^2, y^2, z^2 \rangle$. Note: $\nabla \times \vec{F} = \vec{0}$, so $\vec{F}$ is conservative, so:

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(3,2,0) - f(0,0,0),$$ where f is a scalar potential function for $\vec{F}$.

$f = \int x^2 \partial x$ $\quad$ $f = \int y^2 dy$ $\quad$ and $\quad f = \int z^2 \partial z$

$f = \frac{1}{3}x^3 + g(y,z)$ $\quad$ $f = \frac{1}{3}y^3 + h(x,z)$ $\quad$ $f = \frac{1}{3}z^3 + k(x,y)$

$\Rightarrow \quad f(x,y,z) = \frac{1}{3}(x^3 + y^3 + z^3)$

So, $W = f(3,2,0) - f(0,0,0)$

$= \frac{1}{3}(27 + 8) - 0$

$= \frac{35}{3}$

21)

a) $\vec{F} = \langle x\sin(2x), e^y + e^{-z}, e^y - e^{-z} \rangle$

$\frac{\partial F_1}{\partial y} = 0 \quad \frac{\partial F_2}{\partial x} = 0 \quad \frac{\partial F_1}{\partial z} = 0 \quad \frac{\partial F_3}{\partial x} = 0 \quad \frac{\partial F_2}{\partial z} = -e^{-z} \quad \frac{\partial F_3}{\partial y} = e^y$

Since $\frac{\partial F_2}{\partial z} \neq \frac{\partial F_3}{\partial y}$, $\vec{F}$ is not conservative.

b) $\nabla \cdot \vec{F} = \sin(2x) + 2x\cos(2x) + e^y + e^{-z}$

c) $\nabla \times \vec{F} = \left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle = \langle e^y + e^{-z}, 0, 0 \rangle$

22) $d = \sqrt{x^2 + y^2 + z^2}$ and $\vec{F} = \frac{1}{d^2}\langle x, y, z \rangle$

a) Let $f(x,y,z) = \ln(d) = \frac{1}{2}\ln(x^2 + y^2 + z^2)$

$\nabla f = \frac{1}{2d^2}\langle 2x, 2y, 2z \rangle = \frac{1}{d^2}\langle x, y, z \rangle$

b) $\nabla\cdot\vec{F} = \nabla\cdot\left(\frac{1}{d^2}\langle x,y,z\rangle\right) = \frac{\partial}{\partial x}\frac{x}{d^2} + \frac{\partial}{\partial y}\frac{y}{d^2} + \frac{\partial}{\partial z}\frac{z}{d^2}$

$$= \frac{\partial}{\partial x} x d^{-2} + \frac{\partial}{\partial y} y d^{-2} + \frac{\partial}{\partial z} z d^{-2}$$

$$= d^{-2} - 2xd^{-3}\frac{\partial d}{\partial x} + d^{-2} - 2yd^{-3}\frac{\partial d}{\partial y} + d^{-2} - 2zd^{-3}\frac{\partial d}{\partial z}$$

$$= 3d^{-2} - 2xd^{-3}\left(\frac{\cancel{2}x}{\cancel{2}d}\right) - 2yd^{-3}\left(\frac{\cancel{2}y}{\cancel{2}d}\right) - 2zd^{-3}\left(\frac{\cancel{2}z}{\cancel{2}d}\right)$$

$$= \frac{3}{d^2} - \frac{2}{d^4}\underbrace{\left(x^2+y^2+z^2\right)}_{= d^2}$$

$$= \frac{3-2}{d^2}$$

$$= \frac{1}{d^2}$$

$$= \frac{1}{x^2+y^2+z^2}$$

c) $\nabla\times\vec{F} = \nabla\times\frac{1}{d^2}\langle x,y,z\rangle = \nabla\times\left\langle xd^{-2}, yd^{-2}, zd^{-2}\right\rangle$

$$= \left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right\rangle$$

$$= \left\langle -2zd^{-3}\frac{\partial d}{\partial y} - (-2y)d^{-3}\frac{\partial d}{\partial z}, -2xd^{-3}\frac{\partial d}{\partial z} - (-2z)d^{-3}\frac{\partial d}{\partial x}, -2yd^{-3}\frac{\partial d}{\partial x} - (-2x)d^{-3}\frac{\partial d}{\partial y}\right\rangle$$

$$= -2d^{-3}\left\langle z\frac{\partial d}{\partial y} - y\frac{\partial d}{\partial z}, x\frac{\partial d}{\partial z} - z\frac{\partial d}{\partial x}, y\frac{\partial d}{\partial x} - x\frac{\partial d}{\partial y}\right\rangle$$

$$= \frac{-\cancel{2}d^{-3}\left\langle z(\cancel{2y}) - y(\cancel{2z}), x(\cancel{2z}) - z(\cancel{2x}), y(\cancel{2x}) - x(\cancel{2y})\right\rangle}{\cancel{2}d}$$

$$= \langle 0, 0, 0 \rangle$$

23) $\int_C \vec{F} \cdot d\vec{r}$ is a vector line integral.

$\int_C (\nabla \cdot \vec{F})\, d\vec{r}$ is nonsense. (Since $\nabla \cdot \vec{F}$ is a scalar).

$\int_C (\nabla \cdot \vec{F})\, ds$ is a scalar line integral.

$\int_C (\nabla \times \vec{F}) \cdot d\vec{r}$ is a vector line integral.

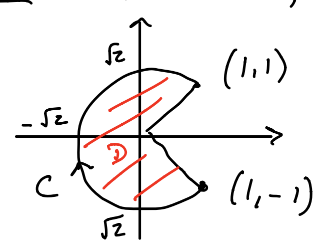
24)

a) $\vec{F} = \langle y, x \rangle$. <u>Note:</u> $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0$ thus $\vec{F}$ is conservative, with a scalar potential function of: $f(x,y) = xy$.

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(1,1) - f(1,-1) = 1 - (-1) = 2$$

b) Note that $\vec{F} = \langle -y, x \rangle$ is <u>not</u> conservative, since

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1 - (-1) = 2.$$



Easier to use Green's Theorem:

$$\int_C \vec{F} \cdot d\vec{r} = \int_{-\partial D} \vec{F} \cdot d\vec{r}$$

negative because Green's requires that the boundary be oriented in such a way that the region lies to the left when traversed. But C is oriented in the opposite fashion.

$$= -\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA$$

$$= -\iint_D 2\, dA$$

$$= -2 \iint_D 1\, dA$$

$$= -2\, \text{Area}(D)$$

$$= -2\left(\frac{3}{4}\right)\pi\left(\sqrt{2}\right)^2$$

$$= -3\pi$$

Alternatively, without using Green's Theorem, we can parameterize C:

$\vec{r}(t) = \langle \sqrt{2}\cos t, -\sqrt{2}\sin t \rangle$ for $t \in \left[\frac{\pi}{4}, \frac{7\pi}{4}\right]$.

$$\vec{r}\,'(t) = \left\langle -\sqrt{2}\sin t,\ -\sqrt{2}\cos t \right\rangle$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{\frac{\pi}{4}}^{\frac{7\pi}{4}} \left\langle \sqrt{2}\sin t,\ \sqrt{2}\cos t \right\rangle \cdot \left\langle -\sqrt{2}\sin t,\ -\sqrt{2}\cos t \right\rangle dt$$

$$= \int_{\frac{\pi}{4}}^{\frac{7\pi}{4}} \left(-2\sin^2 t - 2\cos^2 t\right) dt$$

$$= -2 \int_{\frac{\pi}{4}}^{\frac{7\pi}{4}} dt$$

$$= -2\left(\frac{7\pi}{4} - \frac{\pi}{4}\right)$$

$$= -\frac{12\pi}{4}$$

$$= -3\pi$$

25) $$\int_C \nabla f\cdot d\vec{r} = f(3,3) - f(-1,0) = e^{18} - e$$

26) $$\vec{F} = \left\langle \frac{y}{1+(xy)^2},\ \frac{x}{1+(xy)^2} \right\rangle$$

a) $$\frac{\partial F_2}{\partial x} = \frac{1+(xy)^2 - x\,2xyy}{(1+(xy)^2)^2} \qquad \frac{\partial F_1}{\partial y} = \frac{1+(xy)^2 - y\,2xyx}{(1+(xy)^2)^2}$$

$\Rightarrow \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0 \Rightarrow \vec{F}$ is conservative, so there exists a scalar potential function f such that $\nabla f = \vec{F}$.

$$f = \int F_1\,\partial x \qquad\qquad f = \int F_2\,\partial y$$

$$= \int \frac{y}{1+(xy)^2}\,\partial x \qquad\qquad = \int \frac{x}{1+(xy)^2}\,\partial y$$

$$u = xy \qquad\qquad u = xy$$
$$\partial u = y\,\partial x \qquad\qquad \partial u = x\,\partial y$$

$$= \int \frac{1}{1+u^2}\,\partial u \qquad\qquad = \int \frac{1}{1+u^2}\,\partial u$$

$$= \tan^{-1}(u) + g(y) \qquad\qquad = \tan^{-1}(u) + h(x)$$

$$= \tan^{-1}(xy) + g(y) \qquad\qquad = \tan^{-1}(xy) + h(x)$$

$$\Rightarrow f(x,y) = \tan^{-1}(xy)$$

b) $$\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f\cdot d\vec{r} = f(a,a^{-1}) - f(0,0) = \overset{\frac{\pi}{4}}{\cancel{\tan^{-1}}}(aa^{-1}) - \overset{0}{\cancel{\tan^{-1}}}(0)$$
$$= \frac{\pi}{4}$$

27) $\vec{F} = \langle 2x+3y,\ 2y,\ 2z\rangle$

a) <u>Note:</u> $\dfrac{\partial F_2}{\partial x} = 0 \quad \dfrac{\partial F_1}{\partial y} = 3$. Since these partials are not equal, $\vec{F}$ is not conservative.

b) $C:\ \vec{r}(t) = \langle t, t^2, t^3\rangle$ for $t\in[-1,1]$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{-1}^{1} \langle 2t+3t^2,\ 2t^2,\ 2t^3\rangle\cdot\langle 1, 2t, 3t^2\rangle\,dt$$

$$= \int_{-1}^{1}\left(2t+3t^2+4t^3+6t^5\right)dt$$

$$= t^2+t^3+t^4+t^6\Big|_{-1}^{1}$$

$$= 1+1+\cancel{1}+\cancel{1}-(\cancel{1}-\cancel{1}+\cancel{1}+\cancel{1}) = 2$$

28) "Closed curve"

29) $\vec{F} = \langle z\sec^2 x,\ z,\ y + \tan x \rangle$

$\nabla \times \vec{F} = \langle 1-1,\ \sec^2 x - \sec^2 x,\ 0-0 \rangle = \langle 0,0,0 \rangle$

$\Rightarrow$ $\vec{F}$ is conservative.

$f = \int z\sec^2 x\, \partial x$ $\qquad f = \int z\, \partial y$ $\qquad f = \int (y + \tan x)\, \partial z$

$= z\tan x + g(y,z)$ $\qquad = yz + h(x,z)$ $\qquad = yz + z\tan x + k(x,y)$

$\Longrightarrow$ $f(x,y,z) = yz + \tan x$ is a scalar potential function for $\vec{F}$

30) $\vec{C}(0) = \langle 0,0,1 \rangle$ $\qquad \vec{C}(2) = \langle 4, 1, 1 \rangle$

$\int_C \nabla f \cdot d\vec{r} = f(4,1,1) - f(0,0,1)$

$= 16(1)(1) - 0$

$= 16$

31) $\vec{F} = \langle y^2 + 2cxz,\ bxy + cyz,\ cx^2 + y^2 \rangle$

$\vec{0} = \nabla \times \vec{F}$

$= \left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z},\ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x},\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle$

$= \langle 2y - cy,\ 2cx - 2cx,\ by - 2y \rangle$

$\vec{0} = \langle (2-c)y,\ 0,\ (b-2)y \rangle$

$\Rightarrow$ $(2-c)y = 0$ $\qquad (b-2)y = 0$

$\Rightarrow$ $c = 2$ and $b = 2$

32) $\vec{F} = \langle 2x\cos y,\ \cos y - x^2 \sin y,\ z \rangle$

a) $$\nabla \times \vec{F} = \left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z},\ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x},\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle$$

$$= \langle 0 - 0,\ 0 - 0,\ \cancel{-2x\sin y} + \cancel{2x\sin y} \rangle$$

$$= \vec{0} \implies \vec{F} \text{ is conservative.}$$

b) $\vec{r}(t) = \langle t e^t, \pi t, (1+t)^2 \rangle$ for $t \in [0,1]$.

$\vec{r}(0) = \langle 0, 0, 1 \rangle$ and $\vec{r}(1) = \langle e, \pi, 4 \rangle$

Since $\vec{F}$ is conservative, there exists a scalar potential function f such that $\nabla f = \vec{F}$.

$f = \int 2x\cos y\, \partial x \qquad f = \int (\cos y - x^2 \sin y)\, \partial y \qquad f = \int z\, \partial z$

$f = x^2 \cos y + g(y,z) \qquad f = \sin y + x^2 \cos y + h(x,z) \qquad f = \frac{1}{2}z^2 + K(x,y)$

$\implies f(x,y,z) = \sin y + x^2 \cos y + \frac{1}{2} z^2$ is a scalar potential for $\vec{F}$.

So, $$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r}$$

$e \ \pi \ 4 \qquad 0\ 0\ 1$

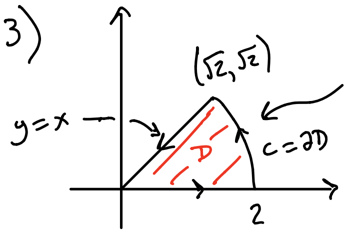
$$= f(\vec{r}(1)) - f(\vec{r}(0))$$

$$= \cancelto{0}{\sin \pi} + e^2 \cancelto{-1}{\cos \pi} + \frac{1}{2}(16) - \left(0 + 0 + \frac{1}{2}\right)$$

$$= -e^2 + 8 - \frac{1}{2}$$

$$= \frac{15}{2} - e^2$$

33)



$x^2+y^2=4$ or $y=\sqrt{4-x^2}$ or $x=\sqrt{4-y^2}$

$$\int_C (x-y^3)\,dx + (x^3-y)\,dy$$

$$\Rightarrow \vec{F} = \langle x-y^3,\ x^3-y \rangle$$

$D: \left.\begin{array}{l} 0 \le \theta \le \frac{\pi}{4} \\ 0 \le r \le 2 \end{array}\right\}$ interpreting D as radially simple.

$$\int_C \vec{F}\cdot d\vec{r} = \int_{\partial D} \vec{F}\cdot d\vec{r}$$

G.T.
$$= \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA$$

$$= \iint_D \left(3x^2 + 3y^2 \right) dA$$

$$= 3\int_{\theta=0}^{\frac{\pi}{4}} \int_{r=0}^{2} r^2 \cdot r\,dr\,d\theta$$

$$= 3\int_0^{\frac{\pi}{4}} d\theta \int_0^2 r^3\,dr$$

$$= \frac{3}{4}\left[\theta \Big|_0^{\frac{\pi}{4}} \right]\left[r^4 \Big|_0^2 \right]$$

$$= \frac{3}{4}\left(\frac{\pi}{4}\right)16$$

$$= 3\pi$$

34) $\vec{F} = \langle 2xy+z,\ x^2+1,\ x+2z \rangle$

$$\nabla \times \vec{F} = \left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z},\ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x},\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle$$

$$= \langle 0-0,\ 1-1,\ 2x-2x \rangle$$

$= \vec{0} \Rightarrow \vec{F}$ is conservative.

Since $\vec{F}$ is conservative, there exists a scalar potential function f such that $\nabla f = \vec{F}$.

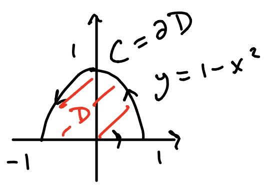
$f = \int (2xy + z)\,\partial x \qquad f = \int (x^2+1)\,\partial y \qquad f = \int (x + 2z)\,\partial z$

$f = x^2 y + xz + g(y,z) \qquad f = x^2 y + y + h(x,z) \qquad f = xz + z^2 + K(x,y)$

$\Longrightarrow f(x,y,z) = x^2 y + xz + y + z^2$ is a scalar potential function for $\vec{F}$.

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(1,1,0) - f(2,-1,1) = 1+1-(-4+2-\cancel{1}+\cancel{1}) = \underline{\underline{4}}$$

35)



$\oint_C -y^2\,dx + xy\,dy \Rightarrow \vec{F} = \langle -y^2, xy \rangle$

$D: \begin{array}{l} -1 \le x \le 1 \\ 0 \le y \le 1-x^2 \end{array}$ } interpreting D as vertically simple.

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_{\partial D} \vec{F}\cdot d\vec{r}$$

$$= \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA$$

$$= \int_{x=-1}^{1} \int_{y=0}^{1-x^2} (y + 2y)\, dy\, dx$$

$$= \int_{x=-1}^{1} \int_{y=0}^{1-x^2} y\, dy\, dx$$

$$= \frac{3}{2} \int_{-1}^{1} y^2 \Big|_0^{1-x^2} dx$$

$$= \frac{3}{2} \int_{-1}^{1} (1-x^2)^2\, dx$$

$$= \frac{3}{2}\int_{-1}^{1}\left(1 - 2x^2 + x^4\right)dx$$

$$= \frac{3}{2}\left[x - \frac{2}{3}x^3 + \frac{1}{5}x^5 \Big|_{-1}^{1}\right]$$

$$= \frac{3}{2}\left[1 - \frac{2}{3} + \frac{1}{5}\right] - \frac{3}{2}\left[-1 + \frac{2}{3} - \frac{1}{5}\right]$$

$$= \frac{3}{2}\left[1 + 1 - \frac{2}{3} - \frac{2}{3} + \frac{1}{5} + \frac{1}{5}\right]$$

$$= \frac{3}{2}\left[2 - \frac{4}{3} + \frac{2}{5}\right]$$

$$= 3\left[1 - \frac{2}{3} + \frac{1}{5}\right]$$

$$= 3 - 2 + \frac{3}{5}$$

$$= 1 + \frac{3}{5}$$

$$= \frac{8}{5}$$