# M273 Final Overview

The formulas in the box will be provided on the exam.

| | | |
|---|---|---|
| $dA = r\,dr\,d\theta$ | $dV = r\,dz\,dr\,d\theta$ | $dV = \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$ |
| $\oint_{\partial\mathcal{D}} P\,dx - Q\,dy = \iint_{\mathcal{D}} (Q_x - P_y)\ dA$ | $\oint_{\partial\mathcal{S}} \mathbf{F}\cdot\,d\mathbf{r} = \oiint_{\mathcal{S}} (\nabla\times\mathbf{F})\cdot\,d\mathbf{S}$ | $\iint_{\partial\mathcal{E}} \mathbf{F}\cdot\,d\mathbf{S} = \iiint_{\mathcal{E}} \nabla\cdot\mathbf{F}\,dV$ |

## §12.1 Vectors in the Plane

- $\mathbf{v} = \langle v_1, v_2\rangle$
- Magnitude of a vector: $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2}$
- Geometric vector addition, "tip-to-tail"
- $\hat{e}_{\mathbf{v}} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ a unit vector which points in the direction of $\mathbf{v}$

## §12.2 Vectors in Three Dimensions

- $\mathbf{v} = \langle v_1, v_2, v_3\rangle$
- Parameterization of a line through the point $(x_0, y_0, z_0)$ with direction vector $\mathbf{v}$

$$\mathbf{r}(t) = \langle x_0, y_0, z_0\rangle + t\mathbf{v}$$

## §12.3 Dot Product and the Angle Between Two Vectors

- Dot product between two vectors $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3\rangle$

$$\mathbf{v}\cdot\mathbf{w} = v_1w_1 + v_2w_2 + v_3w_3$$

- $\mathbf{v}\cdot\mathbf{w} = \|\mathbf{v}\|\,\|\mathbf{w}\|\cos\theta$ where $0 \le \theta \le \pi$ is the angle between the two vectors.
- If $\mathbf{v}\cdot\mathbf{w} = 0$ then $\mathbf{v}$ and $\mathbf{w}$ are orthogonal
- $\mathbf{v}\cdot\mathbf{v} = \|\mathbf{v}\|^2$
- Projection: given two vectors $\mathbf{u}$ and $\mathbf{v} \neq \mathbf{0}$, then projection of $\mathbf{u}$ onto $\mathbf{v}$ is given by

$$\mathbf{u}_{||\mathbf{v}} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}$$

- Orthogonal decomposition: given two vectors $\mathbf{u}$ and $\mathbf{v} \neq \mathbf{0}$ then $\mathbf{u}$ can be written as the sum of two orthogonal vectors

$$\mathbf{u} = \mathbf{u}_{||\mathbf{v}} + \mathbf{u}_{\perp\mathbf{v}}$$

## §12.4 The Cross Product

- Given two vectors $\mathbf{v}$ and $\mathbf{w}$ the cross product is the vector

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

- $\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\| \, \|\mathbf{w}\| \sin\theta$
- Anticommutative: $\mathbf{w} \times \mathbf{v} = -\mathbf{v} \times \mathbf{w}$
- $\mathbf{v} \times \mathbf{w}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$, forming a right-handed system

## §12.5 Planes in 3-Space

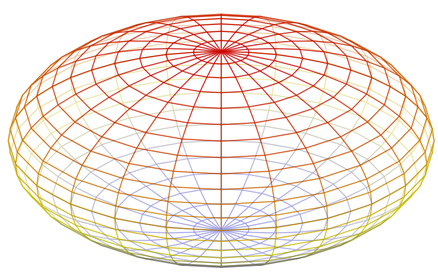
- A plane through a point $(x_0, y_0, z_0)$ with normal vector $\mathbf{n} = \langle a, b, c \rangle$ has equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

## §12.6 A Survey of Quadric Surfaces

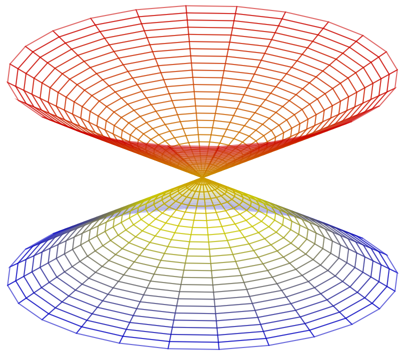
- Ellipsoid

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$$



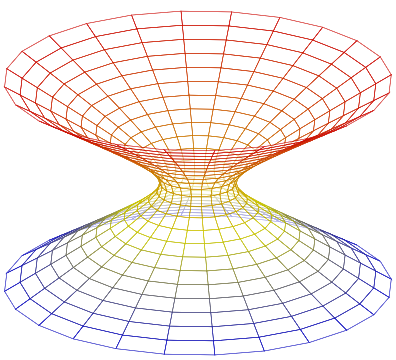
- Elliptic Cone

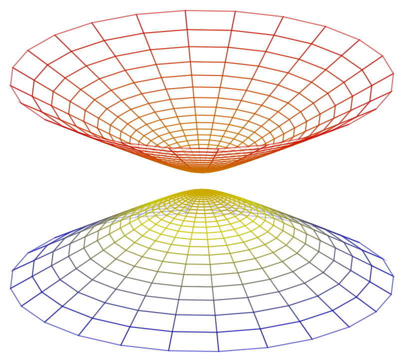
$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \left(\frac{z}{c}\right)^2$$



- Hyperboloids

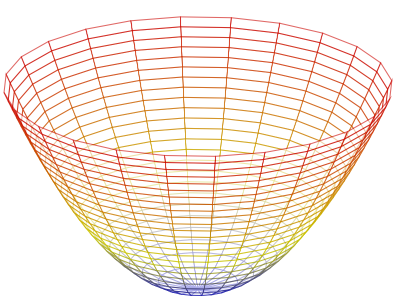
One Sheet: $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \left(\frac{z}{c}\right)^2 + 1$ and Two Sheet: $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \left(\frac{z}{c}\right)^2 - 1$

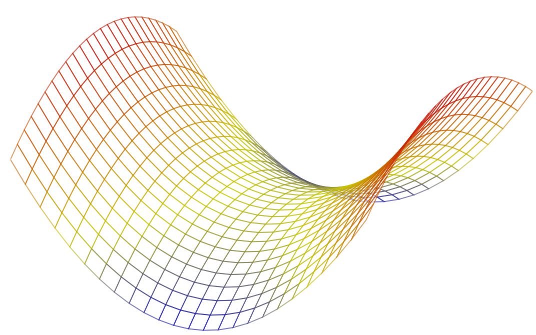




- Paraboloids

Elliptic Paraboloid: $z = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2$ and Hyperbolic Paraboloid: $z = \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2$





## §13.1 Vector-Valued Functions

- A vector-valued function is another name for a parameterization
- $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ is a vector-valued function/parameterization

## §13.2 Calculus of Vector-Valued Functions

- You can perform limits, differentiation, and integration componentwise. e.g. the integral of a vector-valued function is the vector-valued function consisting of the integral of its components.
- Tangent line to $\mathbf{r}(t)$ at $t = t_0$

$$\mathscr{L}(t) = \mathbf{r}(t_0) + t\mathbf{r}'(t_0)$$

## §13.3 Arc Length and Speed

- Speed: given a position/path parameterization $\mathbf{r}(t)$ then the speed is given by $\|\mathbf{r'(t)}\|$
- Arc Length $s$ is the actual distance traveled on a path. The arc length of $\mathbf{r}(t)$ for $t \in [a, b]$ is

$$s = \int_a^b \|\mathbf{r'(t)}\|\; dt$$

- An arc length parameterization of a given parameterization $\mathbf{r}(t)$ is a re-parameterization with the parameter $s$ via the substitution $t = g^{-1}(s)$ which is found by evaluating the arc length integral and finding the inverse (i.e. solving for $t$).

$$s = g(t) = \int_a^t \|\mathbf{r'(u)}\|\; du$$

## §13.4 Curvature

- As stated at the beginning of this overview, if curvature is on the final, you'll be given all of the necessary formulas—you don't have to memorize them. So they are not listed here.
- The unit tangent vector $\mathbf{T} = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r'(t)}\|}$ points in the direction of motion (it's parallel to the velocity vector $\mathbf{r}'(t)$)
- The unit normal vector $\mathbf{N} = \dfrac{\mathbf{T}'(t)}{\|\mathbf{T'(t)}\|}$ points in the direction of bending

## §13.5 Motion in 3-Space

- No new content is presented in this section. It simply applies what has already been taught to more physically-relevant application problems
- The major thing in this section is that the velocity $\mathbf{v}(t) = \mathbf{r}'(t)$ and the acceleration $\mathbf{a}(t) = \mathbf{r}''(t)$

## §14.1 Functions of Two or More Variables

- This is an introductory section on functions of two or more variables
- A function $f(x, y)$ has a domain which is a subset of $\mathbb{R}^2$
- A vertical trace is obtained by intersecting a surface $z = f(x, y)$ with a plane $x = a$ or $y = b$
- A horizontal trace (or level curve) is obtained by intersecting a surface $z = f(x, y)$ with a plane $z = c$
- A contour plot/map is obtained by considering many horizontal traces of a surface $z = f(x, y)$ and projecting all of them onto the $xy$-plane

## §14.2 Limits and Continuity

- If $f(x, y)$ is continuous at $(a, b)$ then $\lim\limits_{(x,y)\to(a,b)} f(x, y) = f(a, b)$ i.e. "just plug it in"
- Limits can be shown to not exist by considering different paths to $(a, b)$
- Common paths to try: $x$-axis, $y$-axis, $y = mx$, and $y = x^2$
- Unlike in Calc I, if any finite number of paths yields the same limiting value, this does not prove the limit exists and equals this value (*all* paths most agree, nearly impossible to check explicitly)

## §14.3 Partial Derivatives

- $f_x(x, y)$ and $f_y(x, y)$ are the first order $x$- and $y$-partials of $f(x, y)$, respectively
- $f_{xy}$ is obtained by first differentiating $f$ with respect to $x$ then with respect to $y$
- $f_{xy} = f_{yx}$ (provided they exist and are continuous)—this generalizes to higher order derivatives

## §14.4 Differentiability and Tangent Planes

- Tangent plane to $z = f(x, y)$ at the point $(a, b)$

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

- Locally to $z = f(a, b)$ the surface can be approximated by the tangent plane, that is

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

## §14.5 The Gradient and Directional Derivatives

- $\nabla = \left\langle \dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z} \right\rangle$
- The gradient of a function $f$

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

- Chain rule for paths: given a function $f(x, y, z)$ and a path parameterized by $\mathbf{r}(t)$ then the rate of change of $f$ along the parameterized path is

$$\frac{d}{dt} f(\mathbf{r}(t)) = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$$

- The rate of change of $f$ in the direction of a unit vector $\mathbf{u}$ is given by

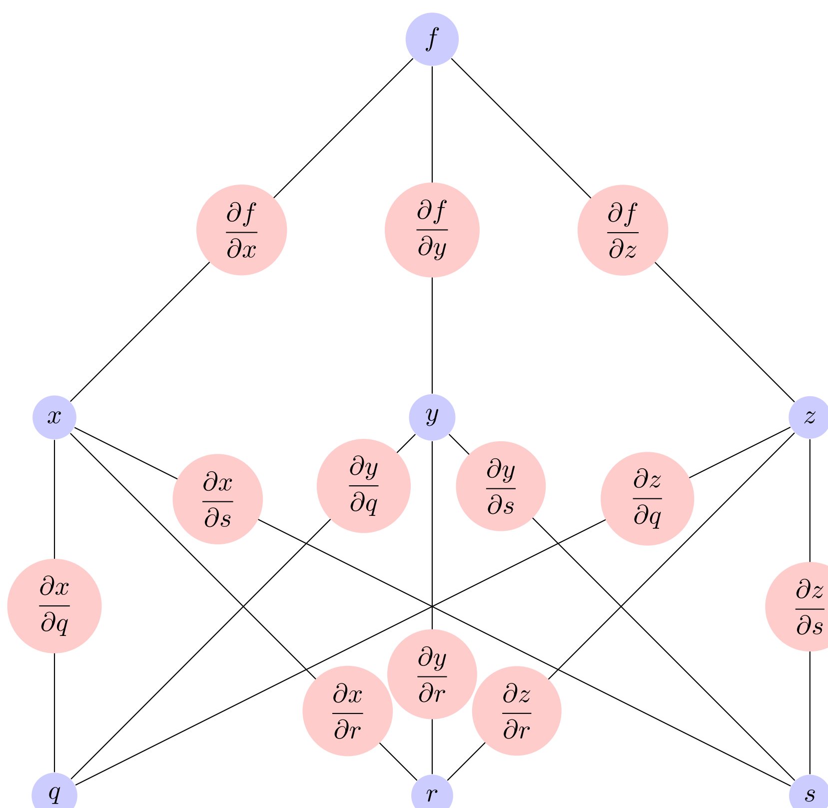
$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$$

- The gradient always points in the direction of the maximum rate of increase
- The negative gradient always points in the direction of the maximum rate of decrease
- The gradient is always orthogonal to the level curves
- Given an equation of the form $F(x, y, z) = 0$, which is an implicit surface, then the gradient $\nabla F$ serves as a normal vector to the surface
- Given the implicit surface $F(x, y, z) = 0$ then the equation of the tangent plane to this surface at the point $(a, b, c)$ is

$$F_x(a, b, c)(x - a) + F_y(a, b, c)(y - b) + F_z(a, b, c)(z - c) = 0$$

## §14.6 The Chain Rule

- Given a function $f(x, y, z)$ where $x = x(q, r, s)$, $y = y(q, r, s)$, and $z = z(q, r, s)$ then a dependency tree can be drawn



Then to write down how $f$ changes with respect to $q$ look at each path from $f$ to $q$ and multiply the partials along each connection and add paths together, yielding

$$\frac{\partial f}{\partial q} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial q} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial q} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial q}$$

- Implicit Differentiation: given an equation of the form $F(x, y, z) = 0$ then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \qquad\qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

## §14.7 Optimization in Several Variables

- Critical Point: A point $(a, b)$ in the domain of $f(x, y)$ is a critical point if $f_x(a, b) = 0$ or DNE AND $f_y(a, b) = 0$ or DNE
- Discriminant: $D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$
- Second Derivative Test: if $(a, b)$ is a critical point of $f$ and all second order partials are continuous at $(a, b)$ then
  - If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum
  - If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum
  - If $D(a, b) < 0$, then $f(a, b)$ is a saddle
  - If $D(a, b) = 0$, then the test is inconclusive
- Finding absolute extrema of $f$ over a domain $\mathcal{D}$:
  - Find local extrema of $f$ that live in $\mathcal{D}$
  - Restrict $f$ to the boundary of $\mathcal{D}$ by performing some form of a substitution
  - Find local extrema of this restricted $f$ using Calc I methods
  - Compare all candidate values: the biggest is the absolute max and the smallest is the absolute minimum

## §14.8 Lagrange Multipliers: Optimizing with a Constraint

- Let $f$ be the function to be optimized (minimized or maximized), this is called the *objective function* subject to the constraint $g(x, y, z) = 0$
- Lagrange Condition:

$$\nabla f = \lambda \nabla g$$

- Find all solutions to the system of equations: $\nabla f = \lambda \nabla g$ and $g(x, y, z) = 0$
- Evaluate $f$ at all solutions to the above system; the highest is the maximum and the lowest is the minimum

## 15.1 Double Integration Over Rectangles

- Fubini's Theorem: integrating over a rectangle $\mathcal{R} = [a,b] \times [c,d]$ can be determined by evaluating an iterated integral in either order,

$$\iint_{\mathcal{R}} f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

## 15.2 Double Integration Over More General Regions

- Vertically simple region $\mathcal{D} : a \leq x \leq b$ and $g_1(x) \leq y \leq g_2(x)$

$$\iint_{\mathcal{D}} f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

- Horizontally simple region $\mathcal{D} : c \leq y \leq d$ and $g_1(y) \leq x \leq g_2(y)$

$$\iint_{\mathcal{D}} f(x,y)\,dA = \int_c^d \int_{g_1(y)}^{g_2(y)} f(x,y)\,dx\,dy$$

- Volume between two surfaces. Let $z = z_1(x,y)$ and $z = z_2(x,y)$ be two surfaces such that $z_1(x,y) \leq z \leq z_2(x,y)$ for all $z \in \mathcal{D}$ where $\mathcal{D}$ is the projection of the bounded region onto the $xy$-plane, then the volume bounded between these two surfaces is given by

$$\iint_{\mathcal{D}} (z_2(x,y) - z_1(x,y))\,dA$$

## 15.4 Part I Double Integration in Polar Coordinates

- Polar conversions: $x = r\cos\theta$ and $y = r\sin\theta$
- Polar differential area element: $dA = r\,dr\,d\theta$
- Radially simple region $\mathcal{D} : \theta_1 \leq \theta \leq \theta_2$ and $r_1(\theta) \leq r \leq r_2(\theta)$

$$\iint_{\mathcal{D}} f(x,y)\,dA = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta) r\,dr\,d\theta$$

## 15.3 Triple Integrals in Cartesian Coordinates

- Fubini's theorem: integrating over a box $\mathcal{B} = [a,b] \times [c,d] \times [p,q]$ can be determined by evaluating an iterated integral in any order (there are $3! = 6$ possible orders)
- Integrating over a $z$-simple region $\mathcal{W} : (x,y) \in \mathcal{D}$ where $z_1(x,y) \leq z \leq z_2(x,y)$ where $\mathcal{D}$ is the projection of $\mathcal{W}$ onto the $xy$-plane:

$$\iiint_{\mathcal{W}} f(x,y,z)\,dV = \iint_{\mathcal{D}} \left( \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz \right) dA$$

- Integrating over a $y$-simple region $\mathcal{W} : (x,z) \in \mathcal{D}$ where $y_1(x,y) \leq y \leq y_2(x,z)$ where $\mathcal{D}$ is the projection of $\mathcal{W}$ onto the $xz$-plane:

$$\iiint\limits_{\mathcal{W}} f(x,y,z)\,dV = \iint\limits_{\mathcal{D}} \left( \int_{y_1(x,z)}^{y_2(x,z)} f(x,y,z)\,dy \right) dA$$

- Integrating over a $x$-simple region $\mathcal{W} : (y,z) \in \mathcal{D}$ where $x_1(y,z) \leq x \leq x_2(y,z)$ where $\mathcal{D}$ is the projection of $\mathcal{W}$ onto the $yz$-plane:

$$\iiint\limits_{\mathcal{W}} f(x,y,z)\,dV = \iint\limits_{\mathcal{D}} \left( \int_{x_1(y,z)}^{x_2(y,z)} f(x,y,z)\,dx \right) dA$$

### 15.4 Part II Triple Integration in Cylindrical and Spherical Coordinates

- Cylindrical conversions: $x = r\cos\theta$, $y = r\sin\theta$, and $z$ remains the same
- Cylindrical differential volume element: $dV = r\,dz\,dr\,d\theta$
- Cylindrically simple region $\mathcal{W} : \theta_1 \leq \theta \leq \theta_2$, $r_1(\theta) \leq r \leq r_2(\theta)$, and $z_1(r,\theta) \leq z \leq z_2(r,\theta)$

$$\iiint\limits_{\mathcal{W}} f(x,y,z)\,dV = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} \int_{z_1(r,\theta)}^{z_2(r,\theta)} f(r\cos\theta, r\sin\theta, z) r\,dz\,dr\,d\theta$$

- Spherical conversions: $x = \rho\cos\theta\sin\phi$, $y = \rho\sin\theta\sin\phi$, and $z = \rho\cos\phi$
- Spherical differential volume element: $dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$
- Spherically simple region $\mathcal{W} : \theta_1 \leq \theta \leq \theta_2$, $\phi_1 \leq \phi \leq \phi_2$, and $\rho_1(\theta,\phi) \leq \rho \leq \rho_2(\theta,\phi)$

$$\iiint\limits_{\mathcal{W}} f(x,y,z)\,dV = \int_{\theta_1}^{\theta_2} \int_{\phi_1}^{\phi_2} \int_{\rho_1(\theta,\phi)}^{\rho_2(\theta,\phi)} f(\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

### Useful Trigonometric Identities

- $\sin^2\theta + \cos^2\theta = 1$
- $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$
- $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$

### 16.1 Vector Fields

- "Nabla" $\nabla = \left\langle \dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z} \right\rangle$
- Divergence operator: $\operatorname{div}(\mathbf{F}) = \nabla \cdot \mathbf{F} = \left\langle \dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z} \right\rangle \cdot \langle F_1, F_2, F_3 \rangle = \dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_2}{\partial y} + \dfrac{\partial F_3}{\partial z}$

- Curl operator: $\operatorname{curl}(\mathbf{F}) = \nabla \times \mathbf{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle$
- A vector field $\mathbf{F}$ is conservative if there exists a scalar potential function $f$ such that $\nabla f = \mathbf{F}$

### 16.2 Line Integrals

- A path $\mathcal{C}$ in $\mathbb{R}^3$ can be parameterized by $\mathbf{r}(t)$ for $t \in [a, b]$
- Scalar line integral of a scalar function $f$ over a path $\mathcal{C}$

$$\int\limits_{\mathcal{C}} f(x, y, z)\, ds = \int_a^b f(\mathbf{r}(t))\, \|\mathbf{r}'(\mathbf{t})\|\ dt$$

- If $f(x, y, z) = 1$ then the scalar line integral of $f$ over a path $\mathcal{C}$ is the length of $\mathcal{C}$, i.e. the arc length,

$$\int\limits_{\mathcal{C}} ds = \int_a^b \|\mathbf{r}'(\mathbf{t})\|\ dt = \text{length}(\mathcal{C})$$

- Vector line integral of a vector field $\mathbf{F}$ over a path $\mathcal{C}$

$$\int\limits_{\mathcal{C}} \mathbf{F} \cdot\, d\mathbf{r} = \int\limits_{\mathcal{C}} (\mathbf{F} \cdot \mathbf{T})\, ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

- Alternate notation for a vector line integral of $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$ over a path $\mathcal{C}$

$$\int\limits_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int\limits_{\mathcal{C}} F_1\, dx + F_2\, dy + F_3\, dz$$

### 16.3 Conservative Vector Fields

- The vector line integral over a closed path (endpoints are equal) $\mathcal{C}$ is called the circulation and is denoted

$$\oint\limits_{\mathcal{C}} \mathbf{F} \cdot\, d\mathbf{r}$$

- If $\mathbf{F}$ is conservative, that is $\mathbf{F} = \nabla f$ for some scalar function $f$, and $\mathcal{C}$ is a path with endpoints $P$ and $Q$ then

$$\int\limits_{\mathcal{C}} \mathbf{F} \cdot\, d\mathbf{r} = \int\limits_{\mathcal{C}} \nabla f \cdot\, d\mathbf{r} = f(Q) - f(P)$$

- If $\mathbf{F}$ is conservative, that is $\mathbf{F} = \nabla f$ for some scalar function $f$, and $\mathcal{C}$ is a closed path, then

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = 0$$

- If $\operatorname{curl}(\mathbf{F}) = \mathbf{0}$ (or if $\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y} = 0$ for 2D fields) and $\mathbf{F}$ is defined on a simply connected domain, then $\mathbf{F}$ is conservative and therefore, there exists a scalar potential function $f$ such that $\nabla f = \mathbf{F}$

- If $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$ is known to be conservative then $f$ can be found by evaluating the following antiderivatives:

$$f = \int F_1 \, dx \qquad\qquad f = \int F_2 \, dy \qquad\qquad f = \int F_3 \, dz$$

## 17.1 Green's Theorem

- Let $\mathcal{D}$ be domain whose boundary $\partial\mathcal{D}$ is a simple closed curve oriented counterclockwise, then

$$\oint_{\partial\mathcal{D}} \mathbf{F} \cdot d\mathbf{r} = \iint_{\mathcal{D}} \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA$$

- When orienting a boundary, if $\mathcal{D}$ lies to the left as the boundary is traversed then this is considered to be oriented positively

- The area of $\mathcal{D}$ can be determined using a line integral around the boundary $\partial\mathcal{D}$ provided

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$$

  A few common fields that have this property are

$$\mathbf{F} = \langle 0, x \rangle \qquad\qquad \mathbf{F} = \langle -y, 0 \rangle \qquad\qquad \mathbf{F} = \left\langle -\frac{y}{2}, \frac{x}{2} \right\rangle$$

- If the boundary of $\mathcal{D}$ is composed of multiple curves, then the total boundary $\partial\mathcal{D}$ can be written as a sum or difference of the constituent curves. For example, if $\partial\mathcal{D}$ is composed of two boundaries $\mathcal{C}_1$ and $\mathcal{C}_2$ then $\partial\mathcal{D} = \pm\mathcal{C}_1 \pm \mathcal{C}_2$ where the choice of plus or minus depends on whether that curve has been oriented positively or negatively. Positively oriented curves get a plus sign and negatively oriented curves receive the minus sign.

## §16.4 Parameterized Surfaces and Surface Integrals

- Sphere: $\mathbf{r}(u, v) = \langle \rho \cos u \sin v, \rho \sin u \sin v, \rho \cos v \rangle$ for $u \in [0, 2\pi]$ and $v \in [0, \pi]$ where $\rho$ is the radius of the sphere

- Hemispheres: use the same parameterization as a sphere but limit the range of $u$ and/or $v$, as appropriate

- Cylinder: $\mathbf{r}(u, v) = \langle r \cos u, r \sin u, v \rangle$ for $u \in [0, 2\pi]$ and $v$ will depend on the cylinder's height and $r$ is the radius

- Cone: assuming it's the cone $z = \sqrt{x^2 + y^2}$ then $\mathbf{r}(u, v) = \langle v \cos u, v \sin u, v \rangle$ for $u \in [0, 2\pi]$ and $v$ will depend on the cone's height

- Function: $z = f(x, y)$ then $\mathbf{r}(u, v) = \langle u, v, f(u, v)\rangle$ where the range of $u$ and $v$ is the same as the range for $x$ and $y$, respectively

- Scalar surface integral of a function $f(x, y, z)$ over a surface $\mathcal{S}$ parameterized by $\mathbf{r}(u, v)$ for $(u, v) \in \mathcal{D}$ then

$$\iint\limits_{\mathcal{S}} f(x, y, z)\, d\mathcal{S} = \iint\limits_{\mathcal{D}} f(\mathbf{r}(u, v))\, \|\mathbf{r_u} \times \mathbf{r_v}\|\; du\, dv$$

- Surface area of a surface $\mathcal{S}$ parameterized by $\mathbf{r}(u, v)$ for $(u, v) \in \mathcal{D}$

$$\text{Area}(\mathcal{S}) = \iint\limits_{\mathcal{S}} dS = \iint\limits_{\mathcal{D}} \|\mathbf{r_u} \times \mathbf{r_v}\|\; du\, dv$$

## §16.5 Surface Integrals of Vector Fields

- Vector surface integral of a vector field $\mathbf{F}$ over a surface $\mathcal{S}$ parameterized by $\mathbf{r}(u, v)$ for $(u, v) \in \mathcal{D}$

$$\iint\limits_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iint\limits_{\mathcal{D}} \mathbf{F}(\mathbf{r}(u, v)) \cdot \mathbf{n}\, du\, dv$$

  where $\mathbf{n}$ is either $\mathbf{r}_u \times \mathbf{r}_v$ or $\mathbf{r}_v \times \mathbf{r}_u$ and will depend on how $\mathcal{S}$ is chosen to be oriented (the problem will say how, somehow)

## §17.2 Stokes' Theorem

- Let $\mathcal{S}$ be a piecewise smooth surface consisting of finitely many positively oriented boundaries, then

$$\iint\limits_{\mathcal{S}} (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint\limits_{\partial\mathcal{S}} \mathbf{F} \cdot d\mathbf{r}$$

- If $\mathbf{F} = \nabla \times \mathbf{A}$ then the flux of $\mathbf{F}$ through a surface $\mathcal{S}$ depends only on the oriented boundary $\partial\mathcal{S}$ and not on the surface itself

$$\iint\limits_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iint\limits_{\mathcal{S}} (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \oint\limits_{\partial\mathcal{S}} \mathbf{A} \cdot d\mathbf{r}$$

  If $\mathcal{S}$ is a closed surface then the boundary of $\mathcal{S}$ is empty, that is $\partial\mathcal{S} = \emptyset$ and so the flux is 0.

## §17.3 Divergence Theorem

- Let $\partial\mathcal{W}$ be a closed surface that encloses a region $\mathcal{W}$ in $\mathbb{R}^3$. Further assume that $\partial\mathcal{W}$ is piecewise smooth and oriented with outward pointing normals (i.e. to the outside of $\mathcal{W}$) then,

$$\iint\limits_{\partial\mathcal{W}} \mathbf{F} \cdot d\mathbf{S} = \iiint\limits_{\mathcal{W}} = \nabla \cdot \mathbf{F}\, dV$$