M273Q Multivariable Calculus
An Old Exam 2

Name and section: Key

Instructor's name: ______

Instructions: Closed book. No calculator allowed. Double-sided exam. NO CELL PHONES. **Show all work and use correct notation to receive full credit!** Write legibly.

1. (2 credit ____) Let $f(x, y, z) = \sin(xyz) - x - 2y - 3z$. Note that

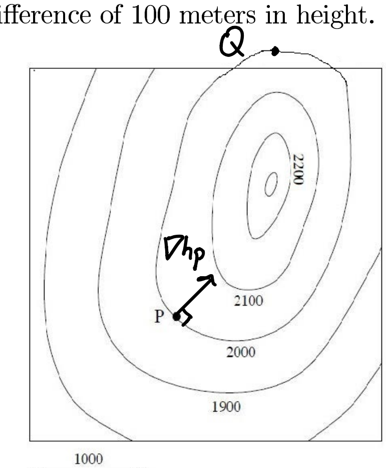
$$\nabla f(x, y, z) = \langle -1 + yz\cos(xyz),\ -2 + xz\cos(xyz),\ -3 + xy\cos(xyz)\rangle\,.$$

Find an equation for the tangent plane to the surface $\sin(xyz) = x + 2y + 3z$ at the point $(2, -1, 0)$.

Let $P = (2, -1, 0)$. Then, $\nabla f_P = \langle -1, -2, -5\rangle$

$$-(x-2) - 2(y+1) - 5z = 0$$

2. On the topographical map below, the level curves for the height function h(x, y) are marked (in meters); adjacent level curves represent a difference of 100 meters in height. A scale is given.



(a) (1 credit ____) At the point P, sketch a vector pointing in the direction of the gradient of h.

(b) (1 credit ____) Mark on the map a point Q at which $h = 2000$, $\dfrac{\partial h}{\partial x} = 0$ and $\dfrac{\partial h}{\partial y} < 0$.

$\dfrac{\partial h}{\partial x} = 0$ means that a small change in the positive x-direction won't change the altitude.

Question:	1	2	Total
Credit	2	2	4
GPA Credit Points Earned			

$\dfrac{\partial h}{\partial y} < 0$ means that a small change in the positive y-direction will decrease the altitude.

3. (2 credit ____) Let

$$w(x,y,z) = xy + yz + zx, \quad x(r,\theta) = r\cos\theta, \quad y(r,\theta) = r\sin\theta, \quad z(r,\theta) = r\theta.$$

Find $\frac{\partial w}{\partial r}$, where $r = 2, \theta = \pi/2$. Let $P = (r,\theta) = (2, \frac{\pi}{2})$. Then,

$x(P) = 0$

$y(P) = 2$

$z(P) = \pi$

$\left.\frac{\partial w}{\partial x}\right|_P = \left. y + z \right|_P = \pi + 2$ $\quad \left.\frac{\partial w}{\partial y}\right|_P = \left. x + z\right|_P = \pi$

$\left.\frac{\partial w}{\partial z}\right|_P = \left. x + y \right|_P = 2$ $\quad \left.\frac{\partial x}{\partial r}\right|_P = \left.\cos\theta\right|_P = 0$

$\left.\frac{\partial y}{\partial r}\right|_P = \left.\sin\theta\right|_P = 1$ $\quad \left.\frac{\partial z}{\partial r}\right|_P = \left.\theta\right|_P = \frac{\pi}{2}$

$$\left.\frac{\partial w}{\partial r}\right|_P = \left. \frac{\partial w}{\partial x}\overset{0}{\cancel{\frac{\partial x}{\partial r}}} + \overset{\pi}{\cancel{\frac{\partial w}{\partial y}}}\overset{1}{\cancel{\frac{\partial y}{\partial r}}} + \overset{2}{\cancel{\frac{\partial w}{\partial z}}}\overset{\frac{\pi}{2}}{\cancel{\frac{\partial z}{\partial r}}}\right|_P = 2\pi$$

Question:	3	Total
Credit	2	2
GPA Credit Points Earned		

4. Evaluate the limit or show that the limit does not exist.

(a) (1 credit ____) $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$ "$\frac{0}{0}$" Approach along the line $y = mx$, where $m \in \mathbb{R}$:

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{x^2+m^2x^2}$$

$$= \frac{1}{1+m^2} \Rightarrow \text{DNE}$$ The limit does not exist

because the value depends on m, the slope of the line. e.g. If $(0,0)$ is approached on the $y=0$, then the answer is 1 but if approached along $y=x$ then the value is $\frac{1}{2}$. Thus, different paths produce different values, i.e. the limit is path-dependent and therefore does not exist.

(b) (1 credit ____) $\lim_{(x,y)\to(1,1)} \frac{4+x-y}{3+x-3y}$ The expression $\frac{4+x-y}{3+x-3y}$ is defined at $(1,1)$.

So, direct substitution:

$$\lim_{(x,y)\to(1,1)} \frac{4+x-y}{3+x-3y} = \frac{4}{1} = 4$$

5. (2 credit ____) Given that $x^3z - 3xy^2 - (yz)^3 = -3$ find $\frac{\partial z}{\partial x}$.

Define $F = x^3z - 3xy^2 - y^3z^3$. Then, $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$

where $F_x = 3x^2z - 3y^2$ and $F_z = x^3 - 3y^3z^2$. Thus,

$$\frac{\partial z}{\partial x} = -\frac{3x^2z - 3y^2}{x^3 - 3y^3z^2}$$

Question:	4	5	Total
Credit	2	2	4
GPA Credit Points Earned			

6. (3 credit ____) Find all critical points of $f(x,y) = x^2 + \frac{1}{3}y^3 - 2xy - 3y$ and classify them (local maximum, local minimum, or saddle) using the Second Derivative Test.

$f_x = 2x - 2y \qquad f_y = y^2 - 2x - 3 \qquad f_{xy} = -2 = f_{yx}$

$f_{xx} = 2 \qquad f_{yy} = 2y \qquad D = 4y - 4 = 4(y-1)$

f_x and f_y are continuous on $\mathbb{R}^2$, so:

$2x - 2y = 0 \qquad y^2 - 2x - 3 = 0$

$y = x \qquad x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0 \Rightarrow x = 3$ or $x = -1$

$\Rightarrow (3,3)$ or $(-1,-1)$.

$D(3,3) = 8 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum at $(3,3)$

$D(-1,-1) = -8 < 0 \Rightarrow$ Saddle at $(-1,-1)$

Question:	6	Total
Credit	3	3
GPA Credit Points Earned		

7. (3 credit ____) Find the coordinates of the points on the ellipse $\frac{x^2}{8} + \frac{y^2}{2} = 1$ at which the function $f(x, y) = xy$ is maximized and those at which f is minimized.

$f(x,y) = xy \leftarrow$ objective function. $g(x,y) = \frac{1}{8}x^2 + \frac{1}{2}y^2 - 1 = 0$ constraint function.

$$\nabla f = \lambda \nabla g$$

$$\langle y, x \rangle = \lambda \langle \tfrac{1}{4}x, y \rangle$$

$$y = \frac{\lambda}{4}x \qquad x = \lambda y \quad \Rightarrow \quad y = \frac{\lambda}{4}(\lambda y)$$

$$y - \frac{\lambda^2}{4}y = 0$$

$$y\left(1 - \frac{\lambda^2}{4}\right) = 0 \Rightarrow y = 0 \text{ or } \lambda = \pm 2.$$

If $y = 0$ then $x = 0$, so $(0,0)$. But $g(0,0) \neq 0$, so this is extraneous.

If $\lambda = \pm 2$ then $y = \pm \frac{1}{2}x$.

If $y = \frac{1}{2}x$ then $\frac{1}{8}x^2 + \frac{1}{8}x^2 = 1$ and so, $x = \pm 2$ and $y = \pm 1$, and therefore: $(\pm 2, \pm 1)$.

If $y = -\frac{1}{2}x$ then $\frac{1}{8}x^2 + \frac{1}{8}x^2 = 1$ and so, $x = \pm 2$ and $y = \mp 1$, and therefore: $(\pm 2, \mp 1)$.

$f(\pm 2, \pm 1) = (\pm 2)(\pm 1) = 2 \Rightarrow$ Absolute maximum of 2 at $(\pm 2, \pm 1)$.

$f(\pm 2, \mp 1) = (\pm 2)(\mp 1) = -2 \Rightarrow$ Absolute minimum of -2 at $(\pm 2, \mp 1)$

Question:	9	10	Total
Credit	1	2	3
GPA Credit Points Earned			

8. Your house lies on the surface $z = f(x, y) = 2x^2 - y^2$ directly above the point $(4, 3)$ in the $xy-$plane.

(a) (1 credit ____) How high above the $xy-$plane do you live?

$$f(4,3) = 32 - 9 = 23$$

(b) (1 credit ____) Calculate the gradient of f at the point $(4, 3)$.

$$\nabla f = \langle 4x, -2y \rangle$$
$$\nabla f(4,3) = \langle 16, -6 \rangle$$

(c) (1 credit ____) What is the slope of your lawn as you look from your house directly toward the $z-$axis (that is, along the vector $< -4, -3 >$)?

Let $\vec{u} = \langle -\frac{4}{5}, -\frac{3}{5} \rangle$

$$D_{\vec{u}} f = \nabla f \cdot \vec{u}$$
$$= \langle 16, -6 \rangle \cdot \langle -\frac{4}{5}, -\frac{3}{5} \rangle$$
$$= \frac{-64 + 18}{5} = -\frac{46}{5}$$

(d) (1 credit ____) When you wash your car in the driveway, on this surface above the point $(4, 3)$, which way does the water run off? (Give your answer as a two-dimensional vector.)

$$-\nabla f(4,3) = -\langle 16, -6 \rangle$$
$$= \langle -16, 6 \rangle$$

Question:	8	Total
Credit	4	4
GPA Credit Points Earned		

9. (1 credit ____) At what point on the surface $z = 1 + x^2 + y^2$ is its tangent plane parallel to the plane $z = 5 + 6x - 10y$?

The plane $z = 5 + 6x - 10y$ has normal vector $\vec{n} = \langle -6, 10, 1 \rangle$.

Define $F(x, y, z) = z - x^2 - y^2$. Then the surface $z = 1 + x^2 + y^2$ is equivalent to the implicit surface $F(x, y, z) = 1$. This surface has normal vector ∇F. If the tangent plane to $F(x, y, z) = 1$ is to be parallel to $z = 5 + 6x - 10y$ then ∇F must be parallel to $\vec{n}$, that is, they must be scalar multiples of one another. Hence,

$$\nabla F = \lambda \vec{n}$$

$$\langle -2x, -2y, 1 \rangle = \lambda \langle -6, 10, 1 \rangle$$

Thus, $\lambda = 1$, and so:

$x = 3$ and $y = -5$ and therefore the point is: $(3, -5, 35)$

$z = 1 + (3)^2 + (-5)^2$

10. Let $f(x, y) = x^7(1 + 2\sin y)$. Note that $f(1, 0) = 1$, $f_x(1, 0) = 7$, and $f_y(1, 0) = 2$.

(a) (1 credit ____) Find an equation of the tangent plane to f at $(1, 0)$.

$$z = 1 + 7(x - 1) + 2y$$

(b) (1 credit ____) Approximate $(0.9)^7(1 + 2sin(0.2))$.

$$(0.9)^7(1 + 2\sin(0.2)) \approx 1 + 7(0.9 - 1) + 2(0.2)$$

$$\approx 1 - 0.7 + 0.4$$

$$\approx 0.7$$

Question	Points	Score
9	1	
10	2	
Total:	3	

Page:	1	2	3	4	5	6	7	Total
Credit	4	2	4	3	3	4	3	23
GPA Credit Points Earned								