Introduction to Analysis II, Spring 2014 Name ______________________
Final Exam. Integration, series, series of functions, multivariate calculus.

Instructions: To prove a theorem, use only the prior results we had at the time the theorem could have first appeared. Do not use results as sophisticated as the given theorem to prove it.
Address conjectures by resolving each with a proof or disproof.

1. (15 pts) Let $S_n = \sum_{k=1}^{n} k^4$. Give a simpler function of n which is asymptotic to S_n for large n.

2. (21 pts) True or False? If it is true, just say so. However, if it is false, you must prove it false.

a) T F If $\sum (a_k)^2$ converges, then $\sum a_k$ converges.

b) T F If f is defined on [0, 1] and has an infinite number of discontinuities, it is not integrable on [0, 1].

c) T F If f is integrable on $[a, b]$, so is f^2.

3. (20 pts) One version of the Fundamental Theorem of Calculus says
$\int_a^b f(t)dt = F(b) - F(a)$ under certain conditions.
a) What are the minimal conditions?
b) Prove the theorem.
[If you don't know how to prove this version, a statement and proof of the other version of the FTC will receive 85% credit.]

4. (8 pts) Short Answer:
a) Which differentiation result corresponds to integration by substitution?

b) Which earlier result yields the formula for the dot product: $\mathbf{u} \bullet \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$.

5. (15 pts) a) What is the definition of $\log(x)$ we gave? [natural log, of course]
b) Use the definition to prove $\log(1/x) = -\log(x)$.

6. (10 pts) Examples
a) Give an example where $\lim \int_a^b f_n(t)dt$ is not $\int_a^b \lim f_n(t)dt$, even though the limits exist.

b) One of the most common examples in power series has an interesting behavior. $\sum_{k=1}^{\infty} a_k x^k$ is continuous for $|x| < R$ where R is its radius of convergence and there is a simple formula for the sum. Even though the formula is continuous at $-R$, the sum does not converge at $-R$. Give the example.

7. (15 pts) Prove (Do not cite radius-of-convergence theorems– they follow this and are not prior):
If $\sum_{k=1}^{\infty} a_k x_0^{\,k}$ converges, then $\sum_{k=1}^{\infty} a_k x^k$ converges uniformly for $|x| \le r < |x_0|$.

8. (20 pts) Suppose we already know that uniform converge of functions makes integrals converge. Use the Fundamental Theorem of Calculus to prove this:
If $f_n(0) = 0$ for all n and $f_n(x)$ converges to $f(x)$ pointwise and $f_n'(x)$ exists and is continuous for all x and converges uniformly to some $g(x)$, then $f_n'(x)$ converges to $f'(x)$.

9. (12 points) a) Define "f_n converges uniformly to f on D."
b) $f_n(x) = nxe^{-nx}$ does not converge uniformly on [0, 1]. Prove it.

10. (12 pts) Here is an equation: $z + z^3 - x^2 + 3xy + y^3 = 46$.
The point (4, 2, 3) solves that equation. Find the equation of the tangent plane which easily yields an approximate solution for z in terms of x and y near (4, 2, 3).

11. (12 pts) Define $f(x, y) = 0$ at (0, 0) and $f(x, y) = \dfrac{xy}{\sqrt{x^2 + y^2}}$ if $(x, y) \neq (0, 0)$.

Prove if it is, or is not, differentiable at (0, 0).

12. (12 pts) Suppose f: $\mathbf{R}^2 \to \mathbf{R}$ is differentiable everywhere and we know all about single-variable calculus and all about directional derivatives. State and prove the two-variable Mean Value Theorem.

13. (12 pts) Suppose f: $\mathbf{R}^2 \to \mathbf{R}$ is differentiable everywhere and so is its derivative and we know the formula for the directional derivative in direction $\boldsymbol{\alpha}$. Derive the second directional derivative in that direction.

14. (16 pts) Find the **exact** sum of

$S_n = \sum_{k=1}^{n} k3^k$ as a simpler function of n. [Note: This is not an infinite series.]