Advanced Calculus, Final Exam on Kosmala Chapters 1-5   Name ____________________
Fall 2014   initial each extra sheet

1. **Definitions**. (12 pts) Define these in sentence-form.
a) accumulation point

b) open set

2. (8 pts) Assume f is defined on $(0, \infty)$. Give the negation of "$\lim_{x\to\infty} f(x) = \infty$".

3. (25 pts) True or False, **no reason required**.

a) T F If $a_n < b_n < 0$ for all n and $\{b_n\}$ is unbounded, then $a_n \to -\infty$

b) T F If f is differentiable on a closed and bounded interval, then it is uniformly continuous.

c) T F If f is uniformly continuous on a bounded open interval, then f is bounded on that interval.

d) T F If $0 \le x_n < 1$ for all n, then $(x_n)^n \to 0$.

e) T F If f' exists for all x and $f'(c) > 0$, then there is a $\delta > 0$ such that $f(c) < f(x)$ for all x in $(c, c + \delta)$.

4. **Examples**. (15 pts) Give an example of each. You do not need to prove that your example works, but make sure it does! Give

a) Functions such that $\lim(f(x)g(x)) = L$, but $(\lim f(x))(\lim g(x))$ is not L.

b) A simple function which is asymptotic to $\dfrac{1}{x + x\sqrt{x+1} + x^2}$ as $x \to 0^+$.

c) A non-linear uniformly continuous function which is unbounded.

5. **Counterexamples**: (12 pts) These conjectures are false. Give a counterexample.

a) Conjecture: If f' exists for all x, then f' is continuous.

b) Conjecture: If f' exists for all x and $f'(c) > 0$, then there is a $\delta > 0$ such that f is increasing on $[c, c + \delta)$.

Proofs and disproofs. Demonstrate that you know how proofs and disproofs are written. **Do not cite similar results to "prove" these**. If a result you want to use is similar to these, or equally difficult, or reminds you of a theorem or example done in class or the book or homework, **do the work again here**. If something you want to use is distinctly prior, use it and **cite it**. Don't argue, prove.

(Do the next eight problems, 16 points each, for a total of 128 points)

6. State and prove either:
Option A) Rolle's Theorem or Option B) The Mean Value Theorem.

7. Prove: Let S be a nonempty set which is bounded above. There exists a sequence $x_n \in S$ such that $x_n \to \sup S$.

8. Prove: If $\lim_{x\to\infty} f(x) = L$ and $x_n \to \infty$, then $\lim f(x_n) = L$.

9. Prove: If f is continuous and $f(a) < k$, then there exists a neighborhood of a in which $f(x) < k$.

10. Resolve this Conjecture: If f and g are differentiable everywhere, $f(a) = g(a)$, and $f'(x) < g'(x)$ for all x, then $f(x) < g(x)$ if $x > a$.

11. Prove: If f is differentiable at c, then f is continuous at c.

12. Prove: If f is uniformly continuous and $\{x_n\}$ is Cauchy, then $\{f(x_n)\}$ is Cauchy.

13. Prove: If $f'(x) \to 4$ as $x \to 0$ and $f'(0)$ exists, then $f'(0) = 4$. [Do not assert f' is continuous--that is not given.]