Name ______________________ Score ______________

Fall 2007

Math 450 Assignment 3

Due Friday, Oct. 19, 2007

1. On pages 32-35 of Logan textbook: Number 8. Here is a guide to how to write up your results.

   (a) List all variables, parameters and their respective dimensions. You are encouraged to use the same notation that was used in class when we discussed the Chemical Reactor example.

   (b) Identify the dimension of k, the rate constant for the assumption that the chemical reaction rate is proportional to the square of the concentration $c(t)$.

   (c) Once you identify the dimension of the rate constant k, you will see that we can no longer nondimensionalize the time variable according to the rate constant k(as we did in the example in class). For the nondimensionalization, use the time scaling and concentration scaling given according to the example in class where

   $$\tau = \frac{t}{V/q} \qquad u(\tau) = \frac{c(t(\tau))}{c_i}.$$

   Begin by giving the governing differential equation in dimensional form, and reformulate the problem in dimensionless form using the given scalings. Your dimensionless form should require the definition of two dimensionless parameters, call them β and γ.

   (d) Show that the dimensionless equation can be expressed in the form

   $$\frac{du}{d\tau} = f(u) = -\beta(u - \lambda_1)(u - \lambda_2),$$

   and explicitly give the form of λ_1 and λ_2 in terms of β.

   (e) Find the equilibrium solutions for the ODE, and classify them as stable or unstable. Discuss the long-term behavior of solutions to the ODE for all relevant initial conditions.

   (f) FOR EXTRA CREDIT! Solve the dimensionless equation explicitly for $u(\tau)$, and show that the long-term behavior of the analytical solution agrees with that of your stability analysis in part (e) above.

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2. On pages 32-35 of Logan textbook: Number 10.

   (a) HINT: In order to identify the appropriate length and time scales for this problem, you will need to first perform a dimensional analysis on the initial value problem. Show that there are 3 dimensionless variables associated with the problem.

   (b) Using the dimensional analysis you just obtained, show that the following scalings can be used to nondimensionalize the problem.

$$\tau = \frac{t}{V/g} \qquad u(\tau) = \frac{h(t(\tau))}{V^2/g}$$

   Give the dimensionless form of the initial value problem.

3. Use **Method of Undetermined Coefficients** to compute the solution to the IVP.

$$u'' + u = 5\cos 3t$$
$$u(0) = 1, \quad u'(0) = 2$$

4. Use **Method of Undetermined Coefficients** to compute the solution to the IVP.

$$u'' + u = -\frac{1}{4}\cos 3t - \frac{3}{4}\sin t$$
$$u(0) = 1, \quad u'(0) = 2$$

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Homework #3

page 32

(8.) Variables + Parameters

(a) t = time . $[t] = T$

$c = c(t)$ = concentration @ time t $[c] = ML^{-3}$

V = constant volume of fluid in Reactor $[V] = L^3$

q = inflow/outflow rate of fluid in tank $[q] = L^3T^{-1}$

c_i = fixed concentration entering reactor $[c_i] = ML^{-3}$

c_0 = initial concentration in reactor $[c_0] = ML^{-3}$

r = reaction rate $[r] = ML^{-3}T^{-1}$

k = constant of proportionality for rate

Assuming the reaction rate is proportional to c^2, we get $r(t) = kc^2$, with k the rate constant

(b.) What are the dimensions of k?

$$r = kc^2$$

$$[r] = [k][c]^2$$

$$[k] = [r] / [c]^2 = \frac{ML^{-3}T^{-1}}{(ML^{-3})^2} = \frac{T^{-1}}{ML^{-3}} = \frac{L^3}{MT}$$

SO, $[k] = L^3M^{-1}T^{-1}$

(C) D.E. via Mass Balance Eqn is:

$$\frac{d}{dt}(Vc(t)) = qc_i - qc - Vkc^2 \quad , \quad c(0)=c_0$$

Scalings:

$$\tau = \frac{t}{V/q} \quad \text{and} \quad u(\tau) = \frac{c(t(\tau))}{c_i}$$

$$\frac{V}{q}\tau = t$$

Note: $\frac{dt}{d\tau} = V/q$ and

$$\frac{du}{d\tau} = \frac{1}{c_i}\frac{dc}{dt}\cdot\frac{dt}{d\tau}$$

$$\frac{du}{d\tau} = \frac{1}{c_i}\left[\frac{q}{V}c_i - \frac{q}{V}c - kc^2\right]\cdot\frac{V}{q}$$

$$\frac{du}{d\tau} = 1 - \frac{c}{c_i} - \frac{kV}{qc_i}c^2$$

$$\frac{du}{d\tau} = 1 - u - \frac{kVc_i}{q}\left(\frac{c}{c_i}\right)^2$$

$$\frac{du}{d\tau} = 1 - u - \frac{kc_iV}{q}u^2 \quad , \quad u(0) = c_0/c_i$$

Define $\beta = \frac{kc_iV}{q}$ and $\gamma = \frac{c_0}{c_i}$.

Then our dimensionless equation becomes

$$\frac{du}{d\tau} = 1 - u - \beta u^2 \quad , \quad u(0) = \gamma$$

(d) Eq. Solns

$$\frac{du}{d\tau} = -\beta u^2 - u + 1$$
$$= -\beta\left(u^2 + \tfrac{1}{\beta}u - \tfrac{1}{\beta}\right)$$
$$= -\beta\,(u-\lambda_1)(u-\lambda_2)$$

where

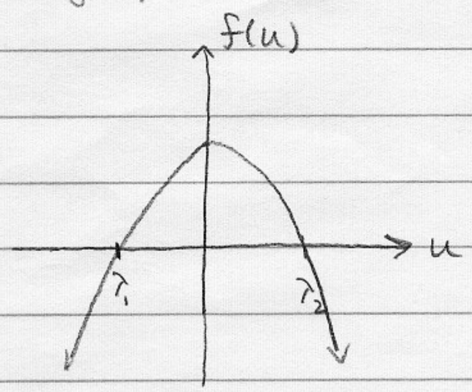
$$\lambda_1 = \frac{-1}{2\beta} - \frac{\sqrt{1+4\beta}}{2\beta}, \quad \lambda_2 = \frac{-1}{2\beta} + \frac{\sqrt{1+4\beta}}{2\beta}$$

Note: $\beta > 0 \Rightarrow \lambda_1 < 0$ and $\lambda_2 > 0$

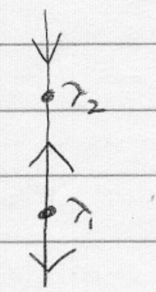
Hence, the equilibrium solns of the ODE are

$$u^*_- = \lambda_1 < 0$$
$$u^*_+ = \lambda_2 > 0$$

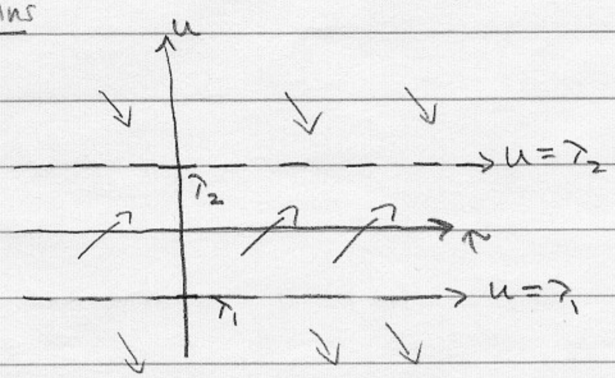
(e). Since $f(u) = -\beta(u-\lambda_1)(u-\lambda_2)$, then the behavior of its graph looks like



Phase Line



Sketch of Solns



$u^* = \lambda_2 = -\frac{1}{2\beta} + \frac{\sqrt{1+4\beta}}{2\beta}$ is a stable equilibrium soln.

Hence, for any initial condition $\gamma \geq 0$, the soln trajectory will approach λ_2. That is

$$\lim_{\tau \to \infty} u(\tau) = \lambda_2 \quad \text{for all } \gamma \geq 0.$$

So

$$\lim_{t \to \infty} \frac{c(t)}{c_i} = -\frac{1}{2\beta} + \frac{\sqrt{1+4\beta}}{2\beta} \quad \text{where } \beta = \frac{k c_i V}{q}$$

(f).

$$\frac{du}{d\tau} = -\beta(u-\lambda_1)(u-\lambda_2)$$

$$\frac{1}{(u-\lambda_1)(u-\lambda_2)} \frac{du}{d\tau} = -\beta$$

$$\int \left[\frac{\frac{1}{\lambda_1-\lambda_2}}{u-\lambda_1} + \frac{\frac{1}{\lambda_2-\lambda_1}}{u-\lambda_2} \right] \frac{du}{d\tau}\, d\tau = \int -\beta\, d\tau$$

$$(\lambda_1-\lambda_2)^{-1} \ln|u-\lambda_1| + (\lambda_2-\lambda_1)^{-1} \ln|u-\lambda_2| = -\beta\tau + K$$

mult. by $(\lambda_2-\lambda_1)$

$$-\ln|u-\lambda_1| + \ln|u-\lambda_2| = -\beta(\lambda_2-\lambda_1)\tau + K(\lambda_1-\lambda_2)$$

$$\ln\left|\frac{u-\lambda_2}{u-\lambda_1}\right| = -\beta(\lambda_2-\lambda_1)\tau + K(\lambda_1-\lambda_2)$$

$$\left|\frac{u-\lambda_2}{u-\lambda_1}\right| = K e^{-\beta(\lambda_2-\lambda_1)\tau}$$

$$\frac{u-\lambda_2}{u-\lambda_1} = Ke^{-\beta(\lambda_2-\lambda_1)\tau}$$

$$(u-\lambda_2) = uKe^{-\beta(\lambda_2-\lambda_1)\tau} - \lambda_1 Ke^{-\beta(\lambda_2-\lambda_1)\tau}$$

$$u(1-Ke^{-\beta(\lambda_2-\lambda_1)\tau}) = \lambda_2 - \lambda_1 Ke^{-\beta(\lambda_2-\lambda_1)\tau}$$

$$u(\tau) = \frac{\lambda_2 - \lambda_1 Ke^{-\beta(\lambda_2-\lambda_1)\tau}}{1-Ke^{-\beta(\lambda_2-\lambda_1)\tau}}$$

Since, $-\beta(\lambda_2-\lambda_1) < 0$, then $\lim_{\tau\to\infty} e^{-\beta(\lambda_2-\lambda_1)\tau} = 0$.

Hence

$$\lim_{\tau\to\infty} u(\tau) = \lambda_2$$

$h>0$

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Homework #3

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(10) <u>Variables & Parameters</u>

t = time $\qquad [t] = T$

$h = h(t)$ = height of ball at time t $\qquad [h] = L$

m = mass of ball $\qquad [m] = M$

V = initial velocity $\qquad [V] = LT^{-1}$

g = gravitational force $\qquad [g] = LT^{-2}$

λ = constant of proportionality $\qquad [\lambda] = ML^{-1}$

- Assume Newton's Law $F=ma$ governs the behavior of the ball.

$$mh''(t) = -mg - \lambda(h'(t))^2$$

$$h(0)=0 \ , \ h'(0) = V$$

<u>Note</u>: $[\lambda(h'(t))^2] = MLT^{-2}$ since $[mh''] = MLT^{-2}$

$$\begin{aligned}
[\lambda] &= MLT^{-2}\,[h'(t)]^{-2} \\
&= MLT^{-2}\,[LT^{-1}]^{-2} \\
&= MLT^{-2}\,L^{-2}T^{2} \\
[\lambda] &= ML^{-1}
\end{aligned}$$

<u>Dimensional Analysis</u>

Let

$$\pi = t^{\alpha_1} h^{\alpha_2} m^{\alpha_3} V^{\alpha_4} g^{\alpha_5} \lambda^{\alpha_6}$$

and require

$$1 = [\pi] = T^{\alpha_1} L^{\alpha_2} M^{\alpha_3} (LT^{-1})^{\alpha_4} (LT^{-2})^{\alpha_5} (ML^{-1})^{\alpha_6}$$

$$1 = T^{\alpha_1 - \alpha_4 - 2\alpha_5}\, L^{\alpha_2 + \alpha_4 + \alpha_5 - \alpha_6}\, M^{\alpha_3 + \alpha_6}$$

$$\left.\begin{aligned} \alpha_1 - \alpha_4 - 2\alpha_5 \qquad &= 0 \\ \alpha_2 + \alpha_4 + \alpha_5 - \alpha_6 &= 0 \\ \alpha_3 + \alpha_6 \qquad &= 0 \end{aligned}\right\} \begin{aligned} \alpha_1 &= \alpha_4 + 2\alpha_5 \\ \alpha_2 &= -\alpha_4 - \alpha_5 + \alpha_6 \\ \alpha_3 &= -\alpha_6 \end{aligned}$$

$$\begin{array}{cc} \begin{array}{c} T \\ L \\ M \\ \end{array} & \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 & -1 & -2 & 0 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \\ \begin{matrix} t & h & m & v & g & \lambda \end{matrix} \end{array} \end{array}$$

Rank = 3 and # of unknowns = 6, so by the Buck Pi. Thm, we have 6 - 3 = 3 dim'less variables. We express the null space of our dim matrix as

$$\vec{\alpha} = \alpha_4 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_5 \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_6 \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \begin{matrix} t \\ h \\ m \\ v \\ g \\ \lambda \end{matrix}$$

One set of dim'less variables are given by

$$\pi_1 = t h^{-1} v \quad , \quad \pi_2 = t^2 h^{-1} g \quad , \quad \pi_3 = h m^{-1} \lambda$$

$$\downarrow \qquad\qquad \downarrow$$

① $h = \frac{1}{\pi_1} t v$ ② $h = \frac{1}{\pi_2} t^2 g$

To obtain a time scaling, use ① + ②

$$\frac{1}{\pi_1} t v = \frac{1}{\pi_2} t^2 g \quad \Rightarrow \quad \frac{\pi_2}{\pi_1} = \frac{t}{v/g} \text{ is dim'less}$$

Linear Algebra Observation

One could obtain the dim'less variable π_2/π_1 (and hence the time scaling $t/\frac{v}{g}$) by choosing

$$\alpha_4 = -1$$
$$\alpha_5 = 1$$
$$\alpha_6 = 0$$

which gives $\alpha_1 = 1$, $\alpha_2 = 0$, $\alpha_3 = 0$ corresponding to

$$\pi_4 = t^1 h^0 m^0 v^{-1} g^1 \lambda^0$$

$$\pi_4 = \frac{tg}{v} = \frac{t}{v/g} \qquad \text{dim'less time scale}$$

In addition, a height scaling can be obtained by choosing

$$\alpha_4 = -2$$
$$\alpha_5 = 1$$
$$\alpha_6 = 0$$

which gives $\alpha_1 = 0$, $\alpha_2 = 1$, $\alpha_3 = 0$ corresponding to

$$\pi_5 = t^0 h^1 m^0 v^{-2} g^1 \lambda^0$$

$$\pi_5 = \frac{h}{v^2/g} \qquad \text{dim'less height scale}$$

- Dim'less Variables

Define

$$\tau = \frac{t}{v/g} \quad \text{and} \quad u(\tau) = \frac{h(t(\tau))}{v^2/g}$$

$$\frac{v}{g}\tau = t \qquad \frac{du}{d\tau} = g/v^{-2}\, \frac{dh}{dt}\cdot\frac{dt}{d\tau} = \frac{1}{v}\frac{dh}{dt}$$

$$\frac{v}{g} = \frac{dt}{d\tau}$$

$$\frac{dt}{d\tau} = \frac{V}{g}, \qquad \frac{du}{d\tau} = \frac{1}{V}\frac{dh(t(\tau))}{dt}$$

$$\frac{d^2u}{d\tau^2} = \frac{d}{d\tau}\left[\frac{1}{V}\frac{dh}{dt}\right] = \frac{1}{V}\frac{d}{dt}\left(\frac{dh}{dt}\right)\cdot\frac{dt}{d\tau} = \frac{1}{g}\frac{d^2h}{dt^2}$$

$$\Rightarrow \boxed{\frac{dh}{dt} = V\frac{du}{d\tau} \quad \text{and} \quad \frac{d^2h}{dt^2} = g\frac{du^2}{d\tau^2}}$$

Begin with

$$m\frac{d^2h}{dt^2} = -mg - \lambda\left(\frac{dh}{dt}\right)^2$$

$$m\left(g\frac{d^2u}{d\tau^2}\right) = -mg - \lambda\left(V\frac{du}{d\tau}\right)^2$$

$$\frac{d^2u}{d\tau^2} = -1 - \frac{\lambda V^2}{mg}\left(\frac{du}{d\tau}\right)^2$$

Define $\beta = \frac{\lambda V^2}{mg}$; also note that $u(0) = \frac{h(0)}{V^2/g} = 0$

$$\frac{du}{d\tau}(0) = \frac{1}{V}\frac{dh(0)}{dt} = \frac{1}{V}(V) = 1$$

<u>Dim'less Eqn</u>:

$$\frac{d^2u}{d\tau^2} = -1 - \beta\left(\frac{du}{d\tau}\right)^2$$

$$u(0)=0, \quad \frac{du}{d\tau}(0) = 1$$

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Homework #3

(3.) Use MUDC's

$$u'' + u = 5\cos(3t) \quad , u(0)=1, \; u'(0)=2$$

Char. Eqn: $r^2+1=0$
$r = \pm i$

$u_h(t) = c_1\cos t + c_2\sin t$

MUDC's Attempt $U(t) = A\cos(3t) + B\sin(3t)$
$U(t) = -3A\sin(3t) + 3B\cos(3t)$
$U(t) = -9A\cos(3t) - 9B\sin(3t)$

$$u'' + u = 5\cos(3t)$$

$$(-9A + A)\cos(3t) + (-9B+B)\sin(3t) = 5\cos(3t)$$

$-8A = 5 \qquad A = -5/8$
$-8B = 0 \qquad B = 0$

$U(t) = -5/8\cos(3t)$

$u(t) = c_1\cos t + c_2\sin t - 5/8\cos(3t)$

ICs $u(0)=1 \rightarrow c_1 - 5/8 = 1 \Rightarrow c_1 = 13/8$
$u'(0)=2 \rightarrow c_2 = 2 \Rightarrow c_2 = 2$

$$u(t) = \frac{13}{8}\cos t + 2\sin t - \frac{5}{8}\cos(3t)$$

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Homework #3

(4) Use MUDC's

$$u'' + u = -\frac{1}{4}\cos(3t) - \frac{3}{4}\sin t$$

$$u(0) = 1, \quad u'(0) = 2$$

As in problem #3, $u_h(t) = C_1\cos t + C_2\sin t$

MUDC's

Attempt: $U(t) = A\cos(3t) + (B\cos t + C\sin t)t$

$$U'(t) = -3A\sin 3t + (B\cos t + C\sin t) + t(-B\sin t + C\cos t)$$

$$U''(t) = -9A\cos 3t + (-B\sin t + C\cos t) + (-B\sin t + C\cos t) + t(-B\cos t - C\sin t)$$

$$U'' = -9A\cos(3t) - 2B\sin t + 2C\cos t + t(-B\cos t - C\sin t)$$

$$U'' + U = -\frac{1}{4}\cos(3t) - \frac{3}{4}\sin t$$

$$-9A\cos(3t) - 2B\sin t + 2C\cos t + t(\cancel{-B\cos t - C\sin t}) + A\cos(3t) + (\cancel{B\cos t + C\sin t})t = -\frac{1}{4}\cos(3t) - \frac{3}{4}\sin t$$

$$-8A\cos(3t) - 2B\sin t + 2C\cos t = -\frac{1}{4}\cos(3t) - \frac{3}{4}\sin t$$

$$-8A = -\frac{1}{4} \qquad -2B = -\frac{3}{4} \qquad 2C = 0$$

$$A = \frac{1}{32} \qquad B = \frac{3}{8} \qquad C = 0$$

$$U(t) = \frac{1}{32}\cos(3t) + \frac{3}{8}t\cos t$$

$$u(t) = C_1\cos t + C_2\sin t + \frac{1}{32}\cos(3t) + \frac{3}{8}t\cos t$$

IC's

$$u(0) = 1 \Rightarrow c_1 + \frac{1}{32} = 1 \Rightarrow c_1 = \frac{31}{32}$$

$$u'(0) = 2 \Rightarrow c_2 + \frac{3}{8} = 2 \Rightarrow c_2 = \frac{13}{8}$$

$$u(t) = \frac{31}{32}\cos t + \frac{13}{8}\sin t + \frac{1}{32}\cos(3t) + \frac{3}{8}t\cos t$$