Name ______________________ Score ________________

Fall 2007

Math 450 Assignment 4

Due Tuesday, Nov. 20, 2007 by 5pm

1. This problem comes from Number 2, page 100 of your textbook. Consider the IVP given below
$$u'' - u = \epsilon t u, \quad u(0) = 1, u'(0) = -1.$$
Find a two-term perturbation approximation for $0 < \epsilon << 1$.

2. On page 101 of Logan textbook: Number 5d.

3. On page 101 of Logan textbook: Number 5g.

4. On page 101 of Logan textbook: Number 8a.

Math 450 - Fall '07
Homework #4
pg. 100, Number 2

(1.) $u'' - u = \varepsilon t u, \quad t > 0$
$u(0) = 1, \; u'(0) = -1$

Rewrite as $u'' - u - \varepsilon t u = 0, \quad t > 0$
$u(0) = 1, \; u'(0) = -1$

- Assume a perturbation series of the form
$u(t) = u_0(t) + \varepsilon u_1(t) + \varepsilon^2 u_2(t) + \cdots$

Algebra:

$$u'' = u_0'' + \varepsilon u_1'' + \varepsilon^2 u_2'' + \cdots$$
$$-u = -u_0 - \varepsilon u_1 - \varepsilon^2 u_2 - \cdots$$
$$-\varepsilon t u = -\varepsilon t u_0 - \varepsilon^2 t u_1 - \varepsilon^3 t u_2 - \cdots$$

Adding + collecting terms

ε^0: $u_0'' - u_0 = 0, \quad u_0(0) = 1, \; u_0'(0) = -1$

ε^1: $u_1'' - u_1 - t u_0 = 0$
$u_1'' - u_1 = t u_0, \; u_1(0) = 0, \; u_1'(0) = 0.$

ε^0: $u_0'' - u_0 = 0$, $u_0(0) = 1$, $u_0'(0) = -1$

$r^2 - 1 = 0$

$r = \pm 1$

$u_0(t) = c_1 e^{-t} + c_2 e^{t}$

$u_0'(t) = -c_1 e^{-t} + c_2 e^{t}$

<u>IC's</u>

$$\begin{aligned} c_1 + c_2 &= 1 \\ -c_1 + c_2 &= -1 \\ \hline 2c_2 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_2 &= 0 \\ c_1 &= 1 \end{aligned}$$

$u_0(t) = e^{-t}$

ε^1: $u_1'' - u_1 = te^{-t}$, $u_1(0) = 0$, $u_1'(0) = 0$. (*)

$u_1(t) = c_1 e^{-t} + c_2 e^{t} + u_p(t)$

Use MUDCs to seek $u_p(t)$ of the form

$$\begin{aligned} u_p(t) &= t(At + B)e^{-t} \\ \rightarrow \quad &= (At^2 + Bt)e^{-t} \end{aligned}$$

$$\begin{aligned} u_p'(t) &= (2At + B)e^{-t} - (At^2 + Bt)e^{-t} \\ \rightarrow \quad &= [-At^2 + (2A - B)t + B]e^{-t} \end{aligned}$$

$$\begin{aligned} u_p''(t) &= (-2At + 2A - B)e^{-t} - (-At^2 + (2A - B)t + B)e^{-t} \\ \rightarrow \quad &= [At^2 + (-4A + B)t + 2A - 2B]e^{-t} \end{aligned}$$

Plugging into (*), we have

$$[At^2 + (B-4A)t + (2A-2B)]e^{-t} - [At^2 + Bt]e^{-t} = te^{-t}$$

$$0t^2e^{-t} + (-4A)te^{-t} + (2A-2B)e^{-t} = te^{-t}$$

$$-4A = 1 \quad \Rightarrow \boxed{A = -\tfrac{1}{4}}$$

$$2A - 2B = 0 \quad \Rightarrow -2B = -2(-\tfrac{1}{4}) = \tfrac{1}{2}$$

$$\boxed{B = -1/4}$$

So, $u_p(t) = \left(-\frac{1}{4}t^2 - \frac{1}{4}t\right)e^{-t} = -\frac{1}{4}t(t+1)e^{-t}$

$$u_1(t) = c_1e^{-t} + c_2e^{t} - \tfrac{1}{4}t(t+1)e^{-t}$$
$$u_1'(t) = -c_1e^{-t} + c_2e^{t} + \tfrac{1}{4}t(t+1)e^{-t} - \tfrac{1}{4}[2t+1]e^{-t}$$

$$u_1(0) = 0 = c_1 + c_2$$
$$u_1'(0) = 0 = -c_1 + c_2 - \tfrac{1}{4}$$

$$c_1 + c_2 = 0$$
$$-c_1 + c_2 = \tfrac{1}{4}$$
$$2c_2 = \tfrac{1}{4} \quad \Rightarrow \quad c_2 = \tfrac{1}{8}, \quad c_1 = -\tfrac{1}{8}$$

$$u_1(t) = -\tfrac{1}{8}e^{-t} + \tfrac{1}{8}e^{t} - \tfrac{1}{4}t(t+1)e^{-t}$$
$$= \tfrac{1}{8}e^{t} - \tfrac{1}{8}e^{-t}[1 + 2t(t+1)]$$

$$u_1(t) = \tfrac{1}{8}e^{t} - \tfrac{1}{8}e^{-t}[1 + 2t + 2t^2]$$

2-term Perturbation Approx.

$$\boxed{u_a(t) = e^{-t} + \varepsilon\left[\tfrac{1}{8}e^{t} - \tfrac{1}{8}e^{-t}(1+2t+2t^2)\right]}$$

Math 450 - Fall '07
Homework #4
pg. 101, Number 5.d

(2.) Show that $\frac{\sqrt{\varepsilon}}{1-\cos(\varepsilon)} = \Theta(\varepsilon^{-3/2})$ as $\varepsilon \to 0^+$

Let $f(\varepsilon) = \sqrt{\varepsilon}/1-\cos(\varepsilon)$ and $g(\varepsilon) = \varepsilon^{-3/2}$

$$\lim_{\varepsilon \to 0} \left| \frac{f(\varepsilon)}{g(\varepsilon)} \right| = \lim_{\varepsilon \to 0} \left| \frac{\varepsilon^{1/2} \cdot \varepsilon^{3/2}}{1-\cos(\varepsilon)} \right|$$

$$= \lim_{\varepsilon \to 0} \left| \frac{\varepsilon^2}{1-\cos(\varepsilon)} \right| \quad \text{has } \tfrac{0}{0} \text{ ind. form}$$

$$\overset{*}{=} \lim_{\varepsilon \to 0} \left| \frac{2\varepsilon}{\sin(\varepsilon)} \right| \quad \text{has } \tfrac{0}{0} \text{ ind. form}$$

$$\overset{*}{=} \lim_{\varepsilon \to 0} \frac{2}{\cos(\varepsilon)} = 2$$

Hence, $f(\varepsilon) = \Theta(g(\varepsilon))$

Math 450 - Fall '07
Homework #4
Pg. 101, Number 5j

(3) Let $f(\varepsilon) = \int_0^{\varepsilon} e^{-x^2}\,dx$ and $g(\varepsilon) = \varepsilon$ as $\varepsilon \to 0^+$.

Note that for $\varepsilon > 0$ and $x \in [0, \varepsilon)$,

$$0 < e^{-x^2} \leq 1$$

So,

$$0 < f(\varepsilon) = \int_0^{\varepsilon} e^{-x^2}\,dx < \int_0^{\varepsilon} 1\,dx = \varepsilon \quad \text{for all } \varepsilon > 0.$$

So,

$$|f(\varepsilon)| = f(\varepsilon) < \varepsilon = |\varepsilon|$$

Hence $f(\varepsilon) = \Theta(\varepsilon)$ as $\varepsilon \to 0^+$.

(82) Use Poincaré-Lindstedt to obtain a 2-term perturbation expansion to $y(t,\varepsilon)$.

(*) $$y''+y=\varepsilon y(y')^2; \quad y(0)=1,\ y'(0)=0.$$

Rescale time according to

* $\tau=\omega t=(\omega_0+\varepsilon\omega_1+\varepsilon^2\omega_2+\cdots)t,$

and assume y has an expansion of the form

* $y(\tau)=y_0(\tau)+\varepsilon y_1(\tau)+\varepsilon^2 y_2(\tau)+\cdots$

Consider $y=y(\tau(t))$, and

* $$\frac{dy}{dt}=\frac{dy}{d\tau}\cdot\frac{d\tau}{dt}=\omega\cdot\frac{dy}{d\tau}$$

* $$\frac{d^2y}{dt^2}=\omega\cdot\frac{d^2y}{d\tau^2}\cdot\frac{d\tau}{dt}=\omega^2\frac{d^2y}{d\tau^2}$$

Plugging this into (*) & denoting $\dot{} = \frac{d}{d\tau}$

$$\omega^2\ddot{y}+y=\varepsilon y(\omega\dot{y})^2$$
$$\omega^2\ddot{y}+y=\varepsilon\omega^2 y(\dot{y})^2$$

* $$\omega^2\ddot{y}+y-\varepsilon\omega^2 y(\dot{y})^2=0$$

* <u>I.C's</u> $y(0)=1=y_0(0)+\varepsilon y_1(0)+\cdots$

$y_0(0)=1,\ y_i(0)=0\ \forall i>1$

* $y'(0)=0=\omega\dot{y}(0) \Rightarrow \dot{y}(0)=0=\dot{y}_0(0)+\varepsilon\dot{y}_1(0)+\cdots$

So, $\dot{y}_i(0)=0\ \forall i=1,2,\ldots$

(82) cont'.

More Scratch:

$$\omega^2 \ddot{y} = (\omega_0 + \varepsilon\omega_1 + \cdots)^2 (\ddot{y}_0 + \varepsilon\ddot{y}_1 + \cdots)$$
$$= (\omega_0^2 + 2\varepsilon\omega_0\omega_1 + \cdots)(\ddot{y}_0 + \varepsilon\ddot{y}_1 + \cdots)$$
$$= \omega_0^2\ddot{y}_0 + \varepsilon(2\omega_0\omega_1\ddot{y}_0 + \omega_0^2\ddot{y}_1) + \cdots$$

$$\omega^2 y(\dot{y})^2 = \omega^2 y(\dot{y}_0 + \varepsilon\dot{y}_1 + \varepsilon^2\dot{y}_2 + \cdots)^2$$
$$= \omega^2 y((\dot{y}_0)^2 + 2\varepsilon\dot{y}_0\dot{y}_1 + \cdots)$$
$$= \omega^2 (y_0 + \varepsilon y_1 + \cdots)((\dot{y}_0)^2 + 2\varepsilon\dot{y}_0\dot{y}_1 + \cdots)$$
$$= \omega^2 (y_0(\dot{y}_0)^2 + \varepsilon(y_1(\dot{y}_0)^2 + 2y_0\dot{y}_0\dot{y}_1) + \cdots)$$
$$= (\omega_0^2 + 2\varepsilon\omega_0\omega_1 + \cdots)[y_0(\dot{y}_0)^2 + \varepsilon(y_1(\dot{y}_0)^2 + 2y_0\dot{y}_0\dot{y}_1) + \cdots]$$
$$= \omega_0^2 y_0(\dot{y}_0)^2 + \varepsilon[2\omega_0\omega_1 y_0(\dot{y}_0)^2 + \omega_0^2 y_1(\dot{y}_0)^2 + 2\omega_0^2 y_0\dot{y}_0\dot{y}_1] + \cdots$$

Plugging these into (*) & collecting terms yields:

ε^0: $\omega_0^2\ddot{y}_0 + y_0 = 0$; $y_0(0) = 1$, $\dot{y}_0(0) = 0$

$\Rightarrow$

$\omega_0^2 r^2 + 1 = 0 \Rightarrow r = \pm i\sqrt{\frac{1}{\omega_0^2}} = \pm i$ since $\omega_0 = 1$

$\ddot{y}_0 + y_0 = 0$

$\Rightarrow \lambda^2 + 1 = 0 \quad \lambda = \pm i$

$y_0(\tau) = c_1\cos(\tau) + c_2\sin(\tau)$

$y_0(0) = 1 = c_1$

$\dot{y}_0(0) = -c_1\sin(0) + c_2\cos(0) = 0 \Rightarrow c_2 = 0$

$\therefore$

$y_0(\tau) = \cos(\tau)$

82 cont.

ε^1: $2\omega_0\omega_1\ddot{y}_0 + \omega_0^2\ddot{y}_1 + y_1 - \omega_0^2 y_0(\dot{y}_0)^2 = 0$

$$\ddot{y}_1 + y_1 - y_0(\dot{y}_0)^2 + 2\omega_1\ddot{y}_0 = 0$$

$$\ddot{y}_1 + y_1 - \cos(\tau)(-\sin(\tau))^2 + 2\omega_1(-\cos(\tau)) = 0$$

$$\ddot{y}_1 + y_1 = \cos(\tau)\sin^2(\tau) + 2\omega_1\cos(\tau)$$

$$= \cos(\tau)(1-\cos^2(\tau)) + 2\omega_1\cos(\tau)$$

$$= (1+2\omega_1)\cos(\tau) - \cos^3(\tau)$$

$$= (1+2\omega_1)\cos(\tau) - \left[\tfrac{1}{4}(3\cos(\tau) + \cos(3\tau))\right]$$

$$\ddot{y}_1 + y_1 = (2\omega_1 + \tfrac{1}{4})\cos(\tau) - \tfrac{1}{4}\cos(3\tau)$$

To eliminate the secular term, we choose ω_1 so that

$$2\omega_1 + \tfrac{1}{4} = 0 \implies \boxed{\omega_1 = -\tfrac{1}{8}}$$

Now, we solve

$$\ddot{y}_1 + y_1 = -\tfrac{1}{4}\cos(3\tau)\ ; \quad y_1(0) = 0, \ \dot{y}_1(0) = 0$$

$y_h(\tau) = A\cos(\tau) + B\sin(\tau)$

$y_p(\tau) = C_1\cos(3\tau) + C_2\sin(3\tau)$

$\dot{y}_p(\tau) = -3C_1\sin(3\tau) + 3C_2\cos(3\tau)$

$\ddot{y}_p(\tau) = -9C_1\cos(3\tau) - 9C_2\sin(3\tau)$

Plugging in:

$$-9C_1\cos(3\tau) - 9C_2\sin(3\tau) + C_1\cos(3\tau) + C_2\sin(3\tau) = -\tfrac{1}{4}\cos(3\tau)$$

$$-8C_1 = -\tfrac{1}{4} \implies \boxed{C_1 = \tfrac{1}{32}}$$

$$-8C_2 = 0 \implies \boxed{C_2 = 0}$$

$y_1(\tau) = A\cos(\tau) + B\sin(\tau) + \frac{1}{32}\cos(3\tau)$

$y_1(0) = 0 = A + \frac{1}{32} \implies \boxed{A = -\frac{1}{32}}$

$\dot{y}_1(0) = 0 = -A\sin(0) + B\cos(0) - \frac{3}{32}\sin(0) \implies \boxed{B = 0}$

$$y_1(\tau) = -\tfrac{1}{32}\cos(\tau) + \tfrac{1}{32}\cos(3\tau)$$

8a. $y_0(\tau) = \cos(\tau)$

$y_1(\tau) = \frac{1}{32}(\cos(3\tau) - \cos(\tau))$

$\omega_0 = 1$ and $\omega_1 = -\frac{1}{8}$

Therefore, our 2-term approx. is given by

* $y_a(\tau) = \cos(\tau) + \frac{1}{32}(\cos(3\tau) - \cos(\tau))\varepsilon$

where

$$\tau = t - \frac{1}{8}\varepsilon t = (1 - \frac{1}{8}\varepsilon)t$$