Name ____________________ Score ____________________

Fall 2007

Math 450 Midterm Exam

Due Wednesday, October 31, 2007

Please provide enough details of your work so that I can follow your reasoning. Without details, I cannot assign partial credit.

1. (25 points) At time $t = 0$ a small amount of chlorine gas, concentrated at a point in space, is allowed to diffuse outward. Suppose that there is a law $f(t, r, m, C, \kappa) = 0$ relating time t, radial distance r from the source, total mass m of the gas, concentration C of the gas, and diffusivity κ of the gas. C has units of mass per unit volume, and κ has units of area per unit time. Each of the quantities t, r, m, C, κ can be expressed in terms of fundamental units of time T, length L, and mass M. You are to

   a. Find the dimension matrix A.

   b. Use the Buckingham Pi Theorem (i.e., linear algebra) to show that there are exactly 2 independent dimensionless variables that can be formed from t, r, m, C, κ.

   c. Show that the 2 independent dimensionless variables can be selected to be

   $$\pi_1 = t^{3/2} m^{-1} C \kappa^{3/2}, \quad \pi_2 = t^{-1/2} r \kappa^{-1/2}.$$

2. (15 points) On page 67 of Logan textbook, Problem 1a.

   a. In addition to the characterization of the equilibrium point $(0, 0)$, sketch the phase plane. Clearly label the solution trajectory satisfying the initial condition $(x(0), y(0)) = (1, 0)$. Assess the long-term behavior of each component of the solution. That is, $x(t) \to ?$ and $y(t) \to ?$ as $t \to \infty$.

   b. Give the solution of the linear system for the initial conditions $(x(0), y(0)) = (1, 0)$. Verify your assertions of the long-term behavior from part a. Generate plots (**using plotting software of your choice**) of $x(t)$ and $y(t)$ for $t > 0$. Plot these on the same axes.

3. (10 points) On page 67 of Logan textbook, Problem 2. Assume that the parameter b is a real number.

4. (15 points) On page 67 of Logan textbook, Problem 4. Note, your sketch in the $pq-$plane should have the $p-$axis as the horizontal axis and the $q-$ axis as the vertical axis.

5. (15 points) On page 79 of Logan textbook, Problem 1e.

6. (20 points) (The directions for this problem are taken from Problem 5, on page 31 of the Logan textbook. However, I am restating the problem and *adjusting* the notation and instructions a little bit.)

The dynamics of a nonlinear mass-spring system is described by

$$mx'' = -ax' - kx^3;$$

$$x(0) = 0, \quad mx'(0) = I,$$

where x is the displacement, $-ax'$ is a linear damping term, and $-kx^3$ is a nonlinear restoring force. Initially, the displacement is zero and the mass m is given an impulse I that starts the motion.

(a) Determine the dimensions of the constants I, a, k.

(b) Perform a dimensional analysis on the problem.

(c) Recast the problem into dimensionless form by selecting dimensionless variables

$$\tau = \frac{t}{Z} \quad \text{and} \quad u = \frac{x}{I/a},$$

where the characteristic time Z is yet to be determined. From your dimensional analysis in part (b), identify one possible characteristic time that one might use for Z.

(d) In the special case where the mass m is very small, choose an appropriate time scaling. In particular, choose a time scale so that once the problem is nondimensionalized, the ODE can be manipulated so that the parameters (including the small parameter m) appear in front of the nonlinear term in the model but the coefficients of $\frac{d^2u}{d\tau^2}$ and $\frac{du}{d\tau}$ do not involve any parameters from the model.

Math 450 Midterm Exam Fall '07

(1) $[t] = T$ $\quad$ Let $\pi = t^{\alpha_1} r^{\alpha_2} m^{\alpha_3} C^{\alpha_4} K^{\alpha_5}$

$[r] = L$

$[m] = M$

$[C] = M/L^3 = ML^{-3}$

$[K] = L^2/T = L^2T^{-1}$

(a)

$$\begin{array}{c} \\ T \\ L \\ M \end{array} \begin{array}{c} \begin{array}{ccccc} t & r & m & C & K \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -3 & 2 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array} = A \quad \left\{ \begin{array}{l} \alpha_3 = -\alpha_4 \\ \alpha_2 = 3\alpha_4 - 2\alpha_5 \\ \alpha_1 = \alpha_5 \end{array} \right.$$

(b) Since A has 3 linearly independent columns, rank(A) = 3. Then by the Buck Pi Thm, there exist exactly 2 independent dimensionless variables. (5-3 = 2)

(c) To find the dimensionless variables, we solve $A\vec{\alpha} = \vec{0}$ where $\vec{\alpha} = [\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4\ \alpha_5]^T$

The matrix in (a) + result in (b) implies that rank(N(A)) = 2. The general soln to $A\vec{\alpha} = \vec{0}$ is given by

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{bmatrix} = \alpha_4 \begin{bmatrix} 0 \\ 3 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \alpha_5 \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{where } \alpha_4, \alpha_5 \in \mathbb{R}^2 \setminus \{\vec{0}\}$$

$$\tilde{\Pi}_1 = t^0 r^3 m^{-1} C K^0 \qquad \text{\&} \qquad \tilde{\Pi}_2 = t r^{-2} m^0 C^0 K$$

$$\tilde{\Pi}_1 = \frac{r^3 C}{m} \qquad \tilde{\Pi}_2 = \frac{tK}{r^2}$$

Note: $\tilde{\Pi}_2^{-1/2} = \dfrac{t^{-1/2} K^{-1/2}}{(r^2)^{-1/2}}$ is also dimensionless.

$\therefore \quad \Pi_2 = t^{-1/2} r K^{-1/2}$ is one dimensionless variables

That is, if we choose the pair $\alpha_4 = 0, \alpha_5 = -1/2$, then Π_3 is the resulting dimensionless quantity.

To obtain Π_1 we solve

$$\alpha_4 \begin{bmatrix} 0 \\ 3 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \alpha_5 \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 0 \\ -1 \\ 1 \\ 3/2 \end{bmatrix}$$

Note: we choose $\alpha_5 = 3/2$ + $\alpha_4 = 1$ by inspection. Hence, this choice of α_4, α_5 gives the dimensionless variable

$$\Pi_1 = t^{3/2} m^{-1} C K^{3/2}$$

And we check lin. indep. of these two

$$c_1 \begin{bmatrix} 3/2 \\ 0 \\ -1 \\ 1 \\ 3/2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{matrix} \Rightarrow \boxed{c_2 = 0} \\ \\ \Rightarrow \boxed{c_1 = 0} \end{matrix}$$

Hence, these two dimensionless variables are indeed independent.

Math 450 - Fall '07
Midterm
p.67 12.

(2)(a) $x' = x - 3y$
$y' = -3x + y$

Define $\vec{u}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, then the system is given by

$$\vec{u}' = \begin{pmatrix} 1 & -3 \\ -3 & 1 \end{pmatrix} \vec{u}$$

E-values: $\det(A - \lambda I) = 0$

$(1-\lambda)(1-\lambda) - 9 = 0$

$\lambda^2 - 2\lambda + 1 - 9 = 0$

$\lambda^2 - 2\lambda - 8 = 0$

$(\lambda - 4)(\lambda + 2) = 0$

$\lambda_1 = 4, \ \lambda_2 = -2$ $\boxed{(0,0) \text{ is a saddle}}$

$\lambda_1 = 4$ $\vec{v}_1 = \begin{pmatrix} x \\ y \end{pmatrix}$ Find x, y so that

$$\begin{pmatrix} 1-4 & -3 \\ -3 & 1-4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{array}{c} -3x - 3y = 0 \\ x = -y \end{array} \quad \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

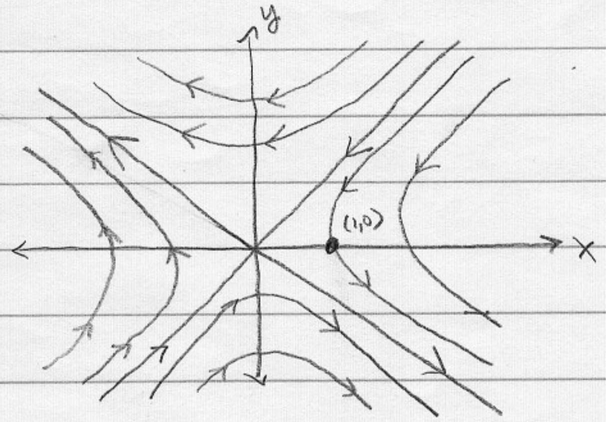
$\boxed{\left(4, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) \text{ is an e-pair}}$

$\lambda_2 = -2$ $\vec{v}_2 = \begin{pmatrix} x \\ y \end{pmatrix}$. Find x, y so that

$$\begin{pmatrix} 1--2 & -3 \\ -3 & 1--2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{array}{c} 3x - 3y = 0 \\ x = y \end{array} \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\boxed{\left(-2, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) \text{ is an e-pair}}$

② (Cont') Phase Plane



For the initial condition $(x(0)=1, y(0)=0)$,
$x(t) \to +\infty$, $y(t) \to -\infty$.

(b) $$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

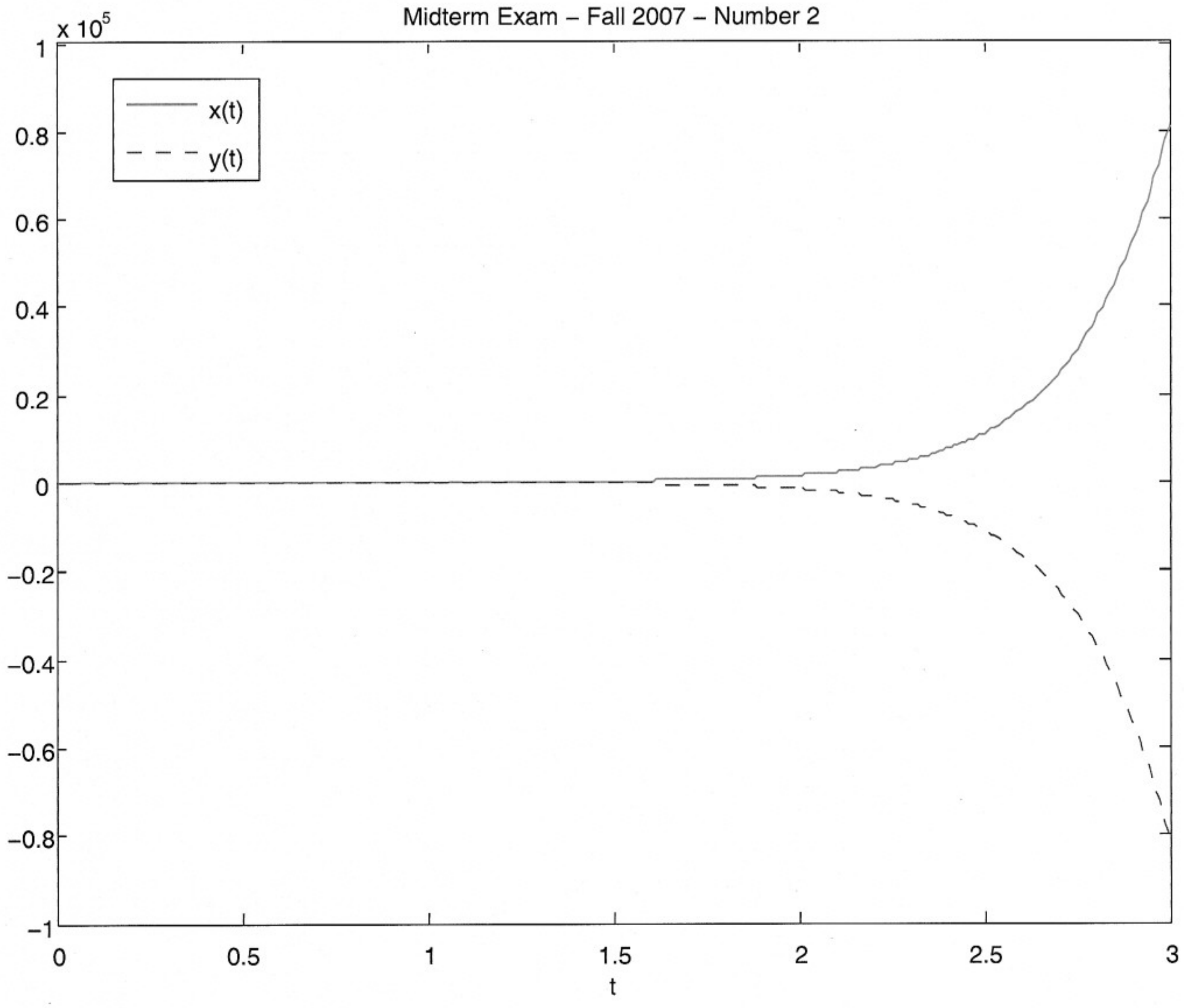
$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$c_1 + c_2 = 1$$
$$-c_1 + c_2 = 0$$
$$2c_2 = 1$$

$\boxed{c_2 = \frac{1}{2} \text{ and } c_1 = \frac{1}{2}}$

$x(t) = \frac{1}{2} e^{4t} + \frac{1}{2} e^{-2t} \to +\infty$ as $t \to \infty$
$y(t) = -\frac{1}{2} e^{4t} + \frac{1}{2} e^{-2t} \to -\infty$ as $t \to \infty$.

Plots $\Rightarrow$


Midterm Exam – Fall 2007 – Number 2
x 10^5
x(t)
y(t)
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
0
0.5
1
1.5
2
2.5
3
t

Math 450 - Fall '07

Midterm

p. 67

(3) $\vec{x}' = \begin{pmatrix} 3 & b \\ 1 & 1 \end{pmatrix} \vec{x}$, let $b \in \mathbb{R}$

Eigenvalues:

$$(3-\lambda)(1-\lambda) - b = 0$$

$$\lambda^2 - 4\lambda + 3 - b = 0$$

$$\lambda = \frac{4 \pm \sqrt{16 - 4(3-b)}}{2}$$

$$\lambda = 2 \pm \tfrac{1}{2}\sqrt{16 - 12 + 4b}$$

$$\lambda = 2 \pm \sqrt{1+b}$$

Several Cases:

(A.) If $b < -1$, then $\lambda = 2 \pm i\sqrt{|1+b|}$

(0,0) is a spiral source

(B) If $b = -1$, then $\lambda = 2$ is a repeated real root that is positive, so (0,0) is a source.

(C) If $-1 < b < 3$, then $\lambda = 2 \pm \sqrt{1+b}$ gives $\lambda_1 = 2 + \sqrt{b} > 0$ and $\lambda_2 = 2 - \sqrt{b} > 0$. Hence, (0,0) is a source.

(d) If $b = 3$, then $\lambda_1 = 2 + \sqrt{4} = 4 > 0$ and $\lambda_2 = 2 - \sqrt{4} = 0$. This will still result in a source in general. Here, we will have a line of pts. that are equilibrium solns.

③ <u>cont'd</u>

(e) If $b>3$, then $\lambda = 2 \pm \sqrt{1+b}$ gives us $\lambda_1 = 2+\sqrt{1+b} > 0$ and $\lambda_2 = 2-\sqrt{1+b} < 0$. Here, $(0,0)$ is a saddle pt.

Math 450 - Fa'07
Midterm
Prob. #4, pg. 67

④ For the Linear System $x' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} x$,
Let $p = tr(A)$ and $q = det(A)$. Shade the region where the origin is a saddle, stable spiral, unstable node, center, etc.?

First, we find the e-values of A by solving

$$(a-\lambda)(d-\lambda) - bc = 0$$

$$\lambda^2 - (a+d)\lambda + ad - bc = 0$$

Notice that $p = tr(A)$ and $q = det(A)$ simplifies this eqn to

$$\lambda^2 - p\lambda + q = 0$$

Using the quadratic formula, the eigenvalues of A are given by

$$\lambda = \frac{p \pm \sqrt{p^2 - 4q}}{2} = \frac{1}{2}p \pm \frac{1}{2}\sqrt{p^2 - 4q} \qquad (*)$$

- Hence, the values of the tr(A) and det(A) determine the stability characteristics of the origin (ie. our equilibrium pt) for the Linear system given above.

- The stability classifications of the origin (0,0) can best be identified by distinguishing the two major cases Ⓐ $p^2 - 4q < 0$ (Complex e-values)
Ⓑ $p^2 - 4q > 0$ (Real e-values)

Note that the parabola $q = \frac{1}{4}p^2$ provides the graph of the boundary between Cases A + B

A) If $p^2 - 4q < 0$ (ie. $q > \frac{1}{4}p^2$), then we have three possibilities

1) If $p^2 - 4q < 0$ AND $p < 0$, then the e-values given in (*) are complex conjugate pairs with $Re(\lambda) = p < 0$. Hence, the origin a "spiral sink" and is "asymptotically stable".

2) If $p^2 - 4q < 0$ AND $p = 0$, then the e-values given in (*) are complex conjugate pairs with $Re(\lambda) = 0$. Hence, the origin is a "Center" and is "stable".

3) If $p^2 - 4q < 0$ AND $p > 0$, then the e-values given in (*) are complex conjugate pairs with $Re(\lambda) = p > 0$. Hence, the origin is a "Spiral Source" and is "unstable".

B) If $p^2 - 4q > 0$ (i.e. $q < \frac{1}{4}p^2$), then we have

4) If $p^2 - 4q > 0$ AND $q < 0$, then $p^2 - 4q > p^2$

Hence, $\lambda = \frac{1}{2}p \pm \frac{1}{2}\sqrt{p^2 - 4q}$

yields two real roots with

$\lambda_1 = \frac{1}{2}p + \frac{1}{2}\sqrt{p^2 - 4q} > 0$ and

$\lambda_2 = \frac{1}{2}p - \frac{1}{2}\sqrt{p^2 - 4q} < 0$

This is because $\sqrt{p^2-4q} > \sqrt{p^2} = |p|$

Hence $\lambda_1 > 0$ and $\lambda_2 < 0$.

Then the origin is a "saddle" point and is unstable.

⑤ <u>If $p^2-4q>0$ AND $q>0$ AND $p>0$</u>, then

$0 < p^2-4q < p^2 \Rightarrow \sqrt{p^2-4q} < \sqrt{p^2} = |p|$

And since $p>0$, then both

$$\lambda_1 = \tfrac{1}{2}p + \tfrac{1}{2}\sqrt{p^2-4q} > 0 \quad \text{and}$$

$$\lambda_2 = \tfrac{1}{2}p - \tfrac{1}{2}\sqrt{p^2-4q} > 0.$$

Hence, the origin is a "Real source" and is unstable.

⑥ If $p^2-4q>0$ AND $q>0$ AND $p<0$, then both

$$\lambda_1 = \tfrac{1}{2}p + \tfrac{1}{2}\sqrt{p^2-4q} < 0 \quad \text{and}$$

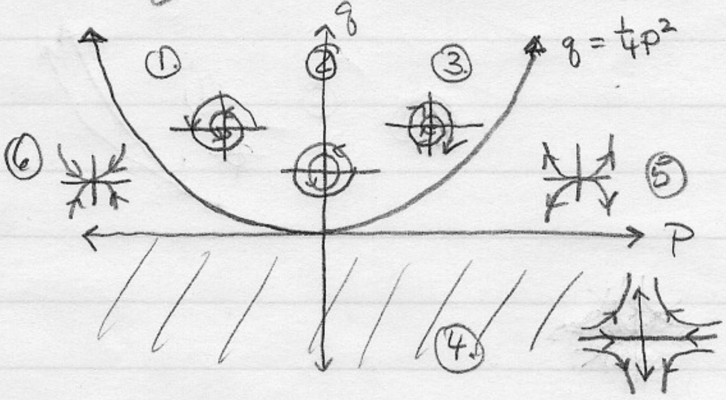
$$\lambda_2 = \tfrac{1}{2}p - \tfrac{1}{2}\sqrt{p^2-4q} < 0.$$

Hence, the origin is a "Real Sink" and is asymptotically stable.

<u>Note</u>: The $q=0$ case as well as the $q=\frac{1}{4}p^2$ cases provided the "boundaries" between the respective behaviors; i.e. where an eigenvalue becomes = zero OR is repeated, respectively.

See Sketch on next page for a geometric perspective.

Diagram of Trace-Determinant Plane



Region ①: "Spiral Sink" → Asymptotically Stable

Region ②: "Center" → Stable

Region ③: "Spiral Source" → Unstable

Region ④: "Saddle" → Unstable

Region ⑤: "Real Source" → Unstable

Region ⑥: "Real Sink" → Asymptotically Stable and

$$\frac{T - \sqrt{T^2 - 4D}}{2} < 0.$$

In this case the system has one positive and one negative eigenvalue, so the origin is a saddle.

In case $T < 0$ and $T^2 - 4D > 0$, we have:

- two negative eigenvalues if $D > 0$
- one negative and one positive eigenvalue if $D < 0$
- one negative eigenvalue and one zero eigenvalue if $D = 0$

Finally, along the repeated-root parabola we have repeated eigenvalues. If $T < 0$, both eigenvalues are negative; if $T > 0$, both are positive; and if $T = 0$, both are zero.

The full picture is displayed in Figure 3.47. Note that this picture gives us some of the same information that we compiled in our table earlier in this section.

Usual Picture Scheme →

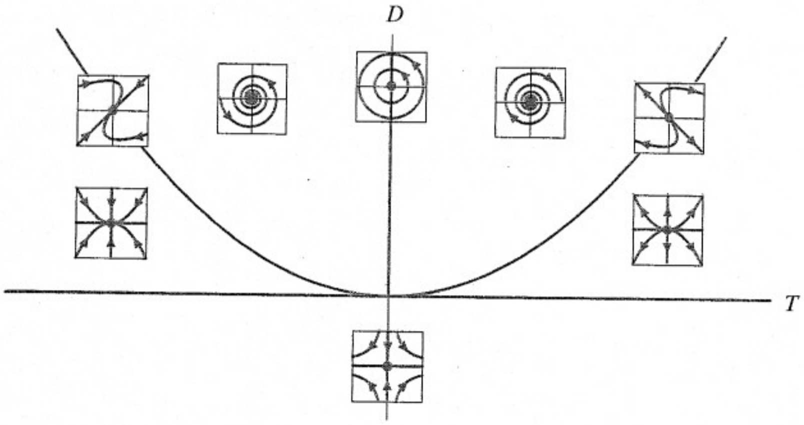


Figure 3.47
The big picture.

The Parameter Plane

The trace-determinant plane is an example of a *parameter plane*. The entries of the matrix **A** are parameters that we can adjust. When these entries change, the trace and determinant of the matrix also change, and our point (T, D) moves around in the parameter plane. As this point enters the various regions in the trace-determinant plane, we should envision the corresponding phase portraits changing accordingly. The trace-determinant plane is very much different from previous pictures we have drawn. It is a picture of a classification scheme of the behavior of all possible solutions to linear systems.

We must emphasize that the trace-determinant plane does not give complete information about the linear system at hand. For example, along the repeated-root parabola we have repeated eigenvalues, but we cannot determine whether we have one or many linearly independent eigenvectors. In order to make that distinction, we must actually calculate the eigenvectors. Similarly, we cannot determine the direction in which solutions wind about the origin if $T^2 - 4D < 0$. For example, both of the matrices

Math 450 - Fall '07
Midterm
Pg. 79, Problem #1e.

⑤ $x' = x^2 + y^2 - 4$
$y' = y - 2x$

Eq pts. Find (x,y) so that
$$x^2 + y^2 - 4 = 0$$
and
$$y - 2x = 0$$

$y = 2x \Rightarrow x^2 + (2x)^2 - 4 = 0$
$$5x^2 = 4$$
$$x^2 = \frac{4}{5}$$
$$x = \pm 2/\sqrt{5}$$

$y = 2x \Rightarrow y = \pm 4/\sqrt{5}$

Two Equilibrium Solns: $\vec{u}_1^* = (2/\sqrt{5}, 4/\sqrt{5})$, $\vec{u}_2^* = (-2/\sqrt{5}, -4/\sqrt{5})$

Jacobian Matrix:
$$F'(x,y) = \begin{pmatrix} 2x & 2y \\ -2 & 1 \end{pmatrix}$$

Examine Linearized System at each equilibrium pt.

u_1^*: $F'(\vec{u}_1^*) = \begin{pmatrix} 4/\sqrt{5} & 8/\sqrt{5} \\ -2 & 1 \end{pmatrix}$

E-values are complex-conjugates with positive real part.
$\lambda_1, \lambda_2 \approx 1.3944 \pm 2.6457i \Rightarrow$ Spiral source (unstable)

$\underline{u_2^*}$: $F'(\vec{u}_2^*) = \begin{pmatrix} -4/\sqrt{5} & -8/\sqrt{5} \\ -2 & 1 \end{pmatrix}$

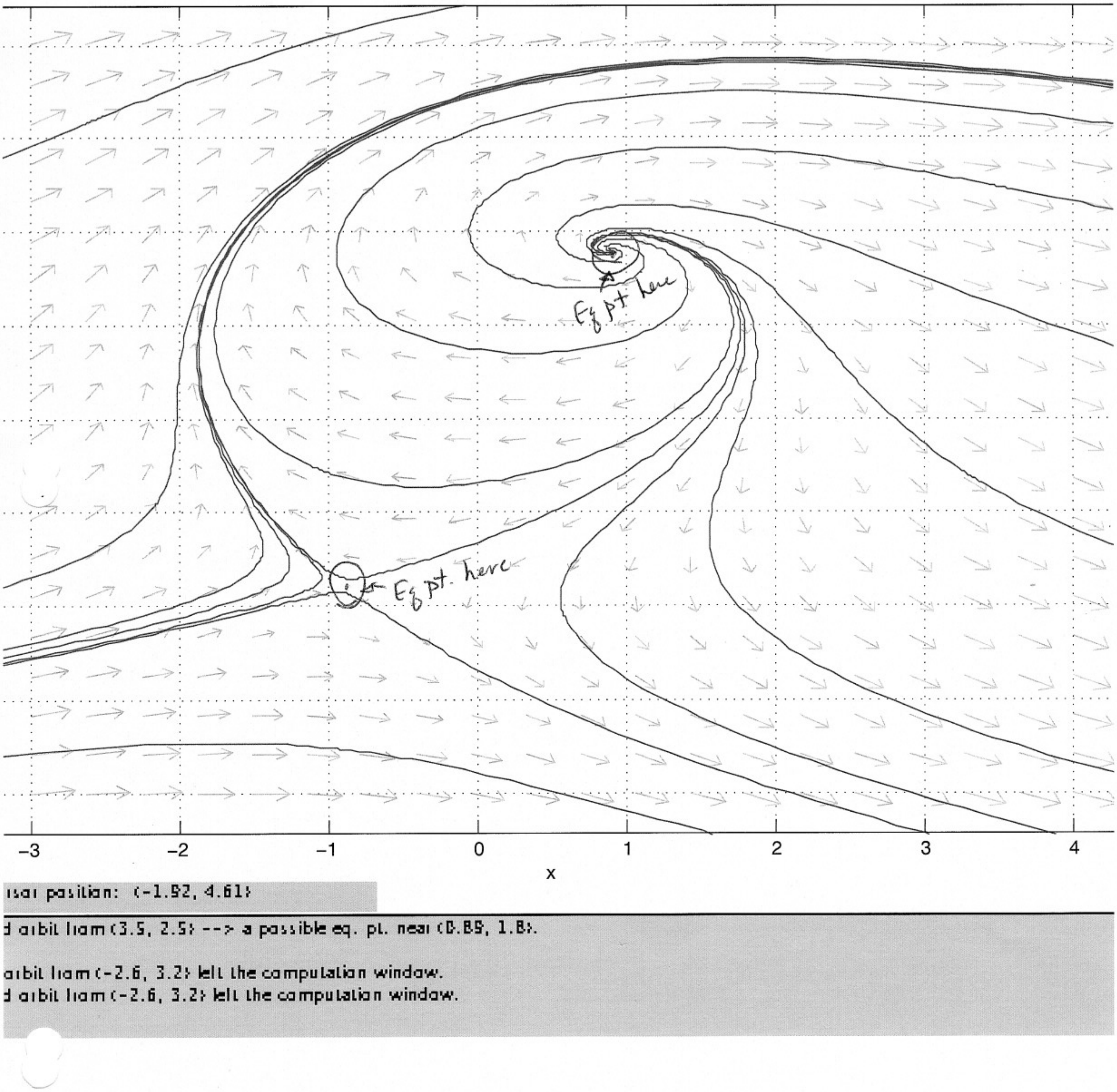
E-val's are real-valued, one positive and one negative.

$\lambda_1 \approx -3.411$ and $\lambda_2 \approx 2.6222$

Hence, u_2^* is a saddle pt. (unstable)

(3)

$= x^2 + y^2 - 4$
$= y - 2x$



ısaı pasilian: (-1.92, 4.61)

d aıbil lıam (3.5, 2.5) --> a passible eq. pt. neaı (0.89, 1.8).

aıbil lıam (-2.6, 3.2) lelt the campulalian window.
d aıbil lıam (-2.6, 3.2) lelt the campulalian window.

Math450 - Fall '07

Midterm

p.31 #5

(1.) t = time $\qquad [t] = T$

$x = x(t)$ = displacement $\qquad [x] = L$

m = mass $\qquad [m] = M$

a = constant of proportionality for damping term $\qquad [a] = MT^{-1}$

k = constant of proportionality for restoring force of spring $\qquad [k] = ML^{-2}T^{-2}$

I = Initial Velocity Constant of proportionality $\qquad [I] = MLT^{-1}$

Governing ODE:

$$mx'' = -ax' - kx^3$$

$$x(0) = 0, \quad mx'(0) = I$$

(2.) Since $[mx''] = [m][x''] = MLT^{-2}$, then the dimensions for each term on the right side must also have these dimensions. This implies

$$[ax'] = MLT^{-2}$$

$$[a][x'] = MLT^{-2}$$

$$[a] = \frac{MLT^{-2}}{[x']} = \frac{MLT^{-2}}{LT^{-1}} = MT^{-1}$$

Similarly,

$$[kx^3] = MLT^{-2}$$

$$[k] = MLT^{-2}/[x]^3 = MLT^{-2}/L^3 = ML^{-2}T^{-2}$$

→

(6a) <u>cont'd</u>

Similarly, from I.C's

$$[mx'] = [I]$$

$$[m][x'] = [I]$$

$$MLT^{-1} = [I]$$

(b) Let $\pi = t^{\alpha_1} x^{\alpha_2} m^{\alpha_3} a^{\alpha_4} k^{\alpha_5} I^{\alpha_6}$

$$1 = [\pi] = T^{\alpha_1} L^{\alpha_2} M^{\alpha_3} M^{\alpha_4} T^{-\alpha_4} M^{\alpha_5} L^{-2\alpha_5} T^{-2\alpha_5} M^{\alpha_6} L^{\alpha_6} T^{-\alpha_6}$$

$$= T^{\alpha_1 - \alpha_4 - 2\alpha_5 - \alpha_6} L^{\alpha_2 - 2\alpha_5 + \alpha_6} M^{\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6}$$

Dimension Matrix

$$\begin{array}{c} \\ T \\ L \\ M \end{array} \begin{array}{c} \begin{array}{cccccc} t & x & m & a & k & I \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & -1 & -2 & -1 \\ 0 & 1 & 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Matrix has rank = 3.

By the Buck.Pi.Thm., there are 3 dimensionless parameters

$$\alpha_1 = \alpha_4 + 2\alpha_5 + \alpha_6$$

$$\alpha_2 = 2\alpha_5 - \alpha_6$$

$$\alpha_3 = -\alpha_4 - \alpha_5 - \alpha_6$$

The soln. can be described by

$$\vec{\alpha} = \alpha_4 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_5 \begin{bmatrix} 2 \\ 2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_6 \begin{bmatrix} 1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

for any choice of $\alpha_4, \alpha_5, \alpha_6$

6 cont'd

c) Possible Dim'less Variables:

$$\Pi_1 = t m^{-1} a = \frac{t}{m/a}$$ ← Possible time Scaling

$$\Pi_2 = t^2 x^2 m^{-1} k$$

$$\Pi_3 = t x^{-1} m^{-1} I = \frac{tI}{xm}$$

Scalings:

$$\tau = \frac{t}{Z} \quad \text{and} \quad u(\tau) = \frac{a\, x(t(\tau))}{I}$$

Note: $\boxed{\frac{dt}{d\tau} = Z}$ and

$$\frac{du}{d\tau} = \frac{a}{I} \cdot \frac{dx}{dt} \cdot \frac{dt}{d\tau}$$

$$\frac{du}{d\tau} = \frac{aZ}{I}\frac{dx}{dt} \quad \text{OR} \quad \boxed{\frac{dx}{dt} = \frac{I}{aZ} \cdot \frac{du}{d\tau}}$$

$$\frac{d^2u}{d\tau^2} = \frac{d}{d\tau}\left[\frac{aZ}{I}\frac{dx(t(\tau))}{dt}\right] = \frac{aZ}{I}\frac{d^2x}{dt^2} \cdot \frac{dt}{d\tau}$$

$$\frac{d^2u}{d\tau^2} = \frac{aZ^2}{I}\frac{d^2x}{dt^2} \quad \text{OR} \quad \boxed{\frac{d^2x}{dt^2} = \frac{I}{aZ^2}\frac{d^2u}{d\tau^2}}$$

Next, we plug these into our original equation

→

(c) cont'd

$$mx'' = -ax' - kx^3$$

$$m\left[\frac{I}{aZ^2}\right]\frac{d^2u}{d\tau^2} = -a\left[\frac{I}{aZ}\frac{du}{d\tau}\right] - k\left[\frac{I}{a}u\right]^3$$

$$\frac{mI}{aZ^2}\frac{d^2u}{d\tau^2} = -\frac{I}{Z}\frac{du}{d\tau} - \frac{kI^3}{a^3}u^3$$

$$\frac{d^2u}{d\tau^2} = \frac{aZ^2}{mI}\left[-\frac{I}{Z}\frac{du}{d\tau} - \frac{kI^3}{a^3}u^3\right]$$

$$\boxed{\frac{d^2u}{d\tau^2} = -\frac{aZ}{m}\frac{du}{d\tau} - \frac{Z^2I^2k}{ma^2}u^3}$$

ICs $\quad u(0) = 0 \quad$ and $\quad m\frac{du}{d\tau}(0) = m\frac{aZ}{I}x'(0) = aZ$

$$\boxed{u(0) = 0 \qquad m\frac{du}{d\tau}(0) = aZ}$$

(d.) If we choose $Z = m/a$ (as in our dimensional analysis), we see that our non-dimensionalized eqn from (c) becomes

$$\boxed{\begin{array}{l} \dfrac{d^2u}{d\tau^2} = -\dfrac{du}{d\tau} - \dfrac{mI^2k}{a^4}u^3 \\[2ex] u(0) = 0, \qquad \dfrac{du}{d\tau}(0) = 1 \end{array}}$$