Name ________________________ Score ____________________

MATH 581 - Fall 2004 - Final Exam

Carefully Read and Follow Directions Clearly label your work and attach it to this sheet. No credit will be given for unsubstantiated answers.

1. Consider the parabolic pde given by

$$\begin{aligned} u_t &= (a(x)u_x)_x, \quad x \in (0,1), t > 0 \\ u(x,0) &= g(x), \quad x \in [0,1] \\ u(0,t) &= u(1,t) = 0, \quad t \geq 0 \end{aligned}$$

and the explicit scheme given by

$$U_j^{n+1} = U_j^n + \nu[a_{j-1/2}U_{j-1} - (a_{j-1/2} + a_{j+1/2})U_j^n + a_{j+1/2}U_{j+1}^n].$$

   (a) Define $a_{\max} = \max\{a(x) \colon 0 \leq x \leq 1\}$, and assume that $a_{\max} > 0$. Also assume that $\nu \leq \frac{1}{2a_{\max}}$. Show that this condition is necessary and sufficient to ensure that the scheme is stable.

   (b) Is this scheme convergent?

2. If the Lax-Friedrichs scheme

$$U_j^{n+1} = \frac{1}{2}(1+\nu)U_{j-1}^n + \frac{1}{2}(1-\nu)U_{j+1}^n, \quad n \geq 0$$

is applied to the one-way wave equation

$$u_t + au_x = 0,$$

   (a) Find the order of the truncation for this method.

   (b) Determine the range of ν for which the method is stable.

3. Consider the PDE

$$u_t + \lambda u_x = 0,$$

and the implicit FD scheme given by

$$(1-R)U_j^{n+1} + RU_{j+1}^{n+1} = U_j^n,$$

where $R = \lambda\frac{\Delta t}{\Delta x}$. Using Discrete Fourier Transform analysis, show that the scheme is stable for $R \leq 0$ or $R \geq 1$.

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4. Now consider the following problem

$$\begin{aligned} u_t + u_x &= 0, \quad x \in (-1, 2), \; t > 0 \\ u(x, 0) &= \begin{cases} 0 & -1 \le x < 0 \\ 1 & 0 \le x \le 1 \\ 0 & 1 < x \le 2 \end{cases} \\ u(-1, t) &= u(2, t), \quad t \ge 0 \end{aligned}$$

with the discontinuous initial condition.

(a) Numerically approximate the solution to this problem using the Lax-Friedrichs scheme (see Problem 2). Use $J = 100$ and $\Delta t = 0.01$. Plot the initial condition and solutions at times $t = 0.05, 1.0$. Use the *subplot* command with a 2×2 matrix of plots. In subplot(2,2,4), include a surface (*surf*) plot of the computed solution over the entire domain of interest.

(b) Repeat part (a) using the Leapfrog scheme. Use the first step of Lax-Friedrichs in order to start up the Leapfrog scheme.