Name ________________________ Score __________________

MATH 581 - Fall 2004 - Homework 1

Carefully Read and Follow Directions Clearly label your work and attach it to this sheet. No credit will be given for unsubstantiated answers.

1. A second order linear partial differential operator $\mathcal{L}$

$$\mathcal{L}u = \sum_{i=1}^{d}\sum_{j=1}^{d} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u$$

is said to be elliptic provided the matrix $A = [a_{ij}]$ has eigenvalues which are all of the same sign (and nonzero). Prove that the operator given by

$$\mathcal{L}u = -\nabla \cdot (\kappa(x)\nabla u), \quad x \in \Omega,$$

is elliptic in Ω provided that there exists a positive constant κ_0 for which $\kappa(x) > \kappa_0$ for all $x \in \Omega$. Note here that we are considering x to represent a vector of space variables in $\mathbb{R}^d$.

2. Consider the parabolic pde given by

$$\begin{aligned} u_t &= u_{xx}, \quad x \in (0,1), t > 0 \\ u(x,0) &= g(x), \quad x \in [0,1] \\ u(0,t) &= u(1,t) = 0, \quad t \geq 0. \end{aligned}$$

If the initial condition $g(x)$ has a Fourier expansion

$$g(x) = \sum_{m=1}^{\infty} a_m \sin(\pi m x),$$

then the solution of the pde can be written explicitly as

$$u(x,t) = \sum_{m=1}^{\infty} \alpha_m e^{-\pi^2 m^2 t} \sin(\pi m x)$$

Consider the following problem, and assume $\epsilon > 0$

$$\begin{aligned} u_t &= \epsilon u_{xx}, \quad x \in (0,1), t > 0 \\ u(x,0) &= \sin 2\pi x, \quad x \in [0,1] \\ u(0,t) &= u(1,t) = 0, \quad t \geq 0. \end{aligned}$$

   (a) Write an explicit statement for the solution to the given pde.

   (b) Give the form of the difference scheme which applies a forward difference in time and a central difference in space for the equation above.

   (c) Write a code (in MatLab) to approximate the solution to the PDE. Using $J = 10$ and $\epsilon = 1/6$, find solutions at $t_F = 0.06$, $t_F = 0.1$, $t_F = 0.9$ and $t_F = 50.0$. For the first three values of t_F, use $\Delta t = 0.02$. Generate plots of the computed solution and plots of the absolute error.